Australian National University

Student Number:

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Mathematical Sciences Institute

EXAMINATION: Semester 2 — Midterm 2022

MATH2322/3104/6118 — Algebra 1

Exam Duration: 120 minutes.
Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- No books, notes, reference materials are permitted.
- No electronic aids are permitted e.g. laptops, phones, calculators.

Materials To Be Supplied To Students:

- Scribble paper (last 3 pages; ask if you need more).

Instructions To Students:

- The exam is worth a total of 60 points, with the value of each question as shown.
- Unless asked otherwise, you must justify your answers. Please be neat and concise.
- You may use any result from class, homework, or workshops as long as it does not trivialise the question.
- You must cite what you use either by name ("Using Lagrange's theorem...") or by recalling the statement ("Since the kernel is a normal subgroup...").
- The last 3 pages are blank. You may detach them and use them for scratch work.

Q1	Q2	Q3	Q4	Q5
16	8	10	10	16

Total / 60

Question 1 ***16 pts***

Give an example or state that no such example exists. *No justification is necessary.*

(a) Two groups of order 6 that are not isomorphic to each other.

(b) A finite non-trivial subgroup of $\mathbf{Z}$.

(c) A non-constant homomorphism $\mathbf{Z}/3\mathbf{Z} \to \mathrm{GL}_2(\mathbf{R})$.

(d) An element of order 6 in $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

(e) A normal subgroup of M_2, the group of isometries of the plane, except $\{e\}$ or M_2.

(f) A surjective homomorphism $\mathbf{Z} \times \mathbf{Z} \to S_3$.

(g) A non-abelian group of order 11.

(h) A non-abelian group of order 12.

Question 2 ***8 pts***

(a) Define the kernel of a homomorphism. Give an example of a homomorphism whose kernel is neither trivial nor the entire group. *No justification is necessary.* *4 pts*

(b) Define the center of a group. Give an example of a group whose center is neither trivial nor the entire group. *No justification is necessary.* *4 pts*

Question 3 ***10 pts***

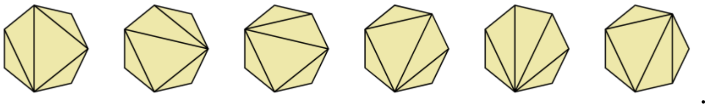
Prove or give a counterexample with justification.

(a) If N is a normal subgroup of G, then G is isomorphic to $N \times G/N$. *5 pts*

(b) If G is a finite abelian group of odd order, then the function $f : G \to G$ defined by $f(x) = x^2$ is an isomorphism. *5 pts*

Question 4 ***10 pts***

A *triangulation* of an n-gon is an unordered collection of non-crossing diagonals that divide the n-gon into triangles. The following picture shows some triangulations of a 7-gon:



Let Σ be the set of triangulations of a regular hexagon. It turns out that Σ has 14 elements. Let $G = D_6$. The standard action of G on the hexagon induces an action on Σ.

(a) Determine the number of G-orbits of Σ. *No justification is necessary.* *3 pts*

(b) For each orbit, draw a triangulation T in that orbit, and find its stabiliser $G_T \subset G$.

Your answer must explicitly state which elements of G are contained in G_T. *No justification is necessary.* [1] *7 pts*

[1] If you do not wish to draw, you may label the vertices by $A_1, \cdots, A_6$ and specify T by listing the diagonals.

Extra space for previous question

Question 5 ***16 pts***

Let G be a group. The *commutator subgroup* H of G is the smallest subgroup that contains the set

$$\{aba^{-1}b^{-1} \mid a, b \in G\}.$$

(a) Prove that H is a *normal* subgroup of G. *6 pts*

(b) True or false: G/H must be abelian. Justify your answer. *4 pts*

(c) Let $G = O_2$. Find H and identify G/H up to isomorphism. Justify your answer. *6 pts*

Extra space for previous question

Scratch work

Scratch work

Scratch work