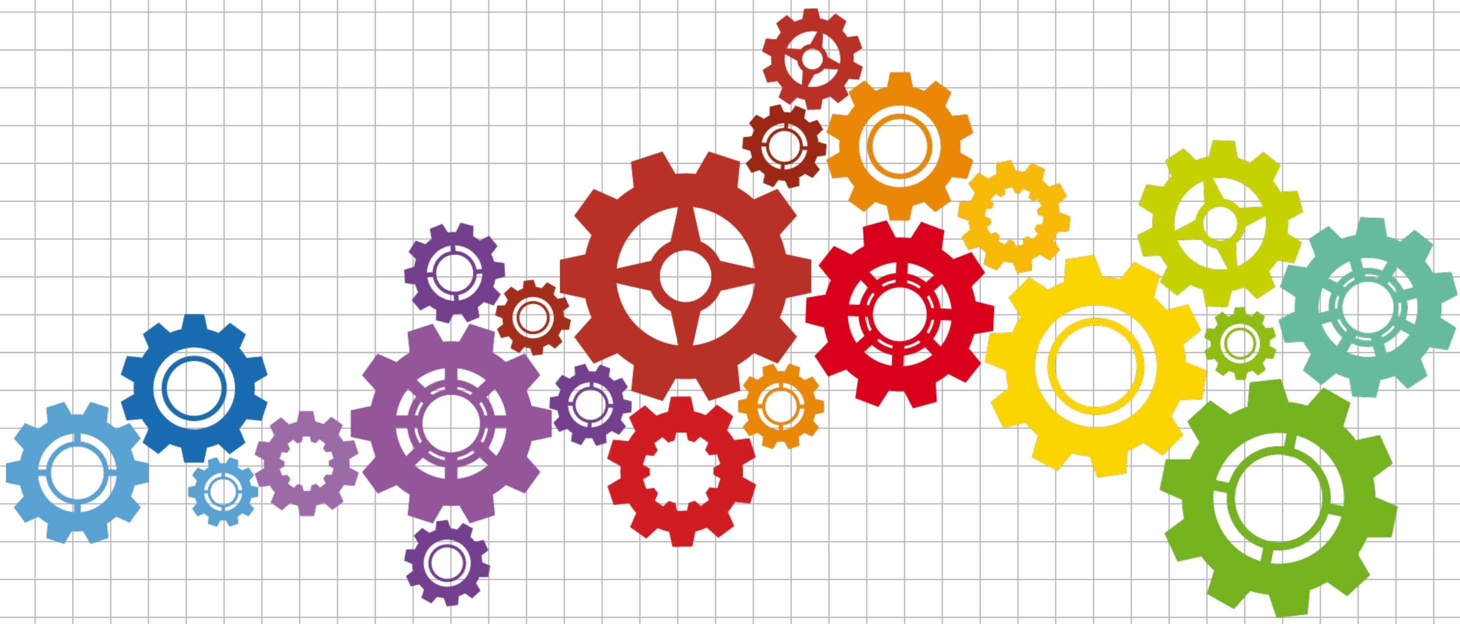
Solvability



Question: Given $f(x)$ can we express roots of f in terms of nested radicals?

Nested radical / surd.

ex. $\sqrt[4]{\sqrt{2} - \sqrt[11]{15 + \sqrt{8}}}$

Precise def: K/F field ext^n.

$\alpha \in K$ is a nested radical over F if there is a chain of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_n$$

with $\alpha \in F_n$ and

$F_{i+1} = F_i[a_i]$ where

some power of $a_i \in F_i$

prime power.

" $F_{i+1} = F_i[\sqrt[m]{b_i}]$, $b_i \in F_i$ "

$\rightarrow \in \mathbb{C}$ is a nested radical over $\mathbb{Q}$.

<u>Q</u>: Given $f(x) \in F[x]$ & a field K in which $f(x)$ splits are the roots nested radicals over F ?

<u>A</u>: Yes for $f(x)$ of deg 2, 3, 4. (char 0).

<u>Thm</u>. (Abel + Galois + Kummer + ...)
There exist $f(x) \in \mathbb{Q}[x]$ of deg 5 (& above) whose complex roots

are not nested radicals over $\mathbb{Q}$.

<u>Folklore</u> :- There is no "formula" for the roots of a quintic.

ingredients: $+, -, \times, \div$

$\sqrt[2]{}, \sqrt[3]{}, \sqrt[4]{}, \sqrt[5]{}, \ldots$

Proof idea - Heart of Galois theory.

$f(x) \rightsquigarrow \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ roots.

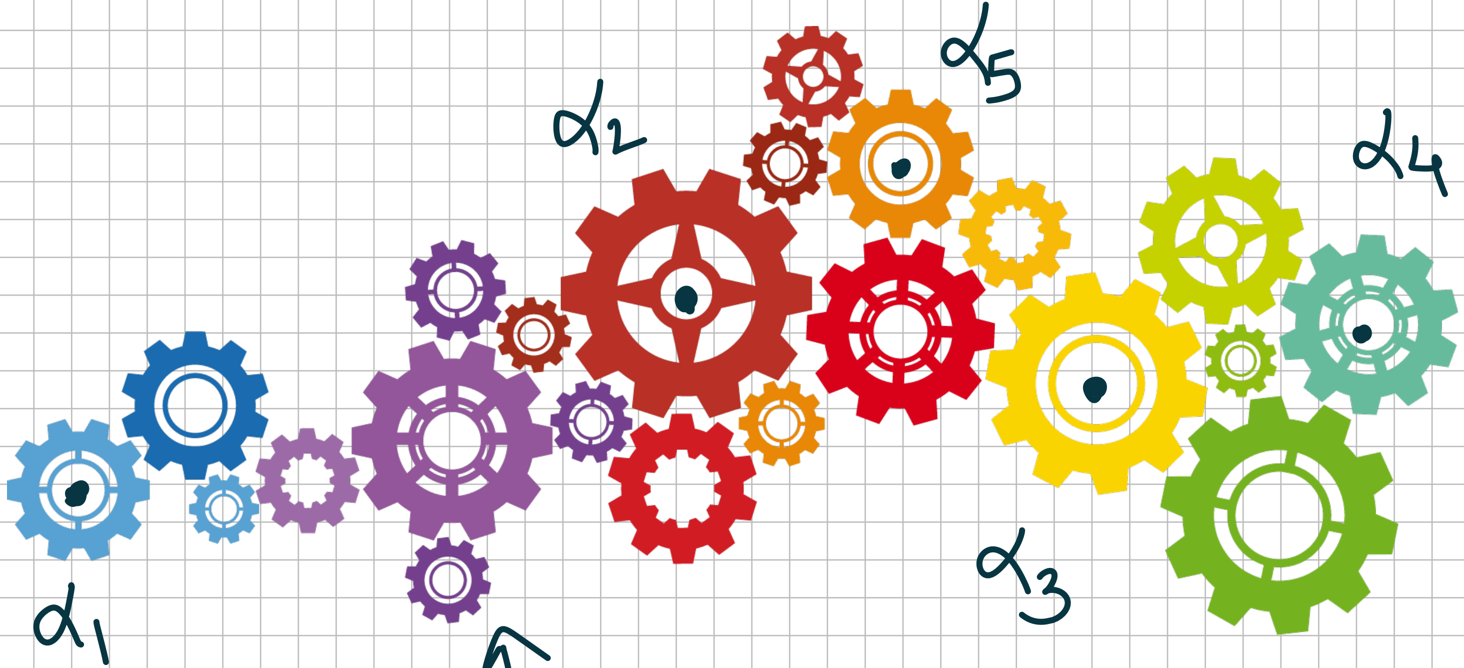
Principle • If α_i have specific alg. structure then (often) there are unexpected alg. relations among them.

$\Updownarrow$

Impediments to permuting the roots coherently.

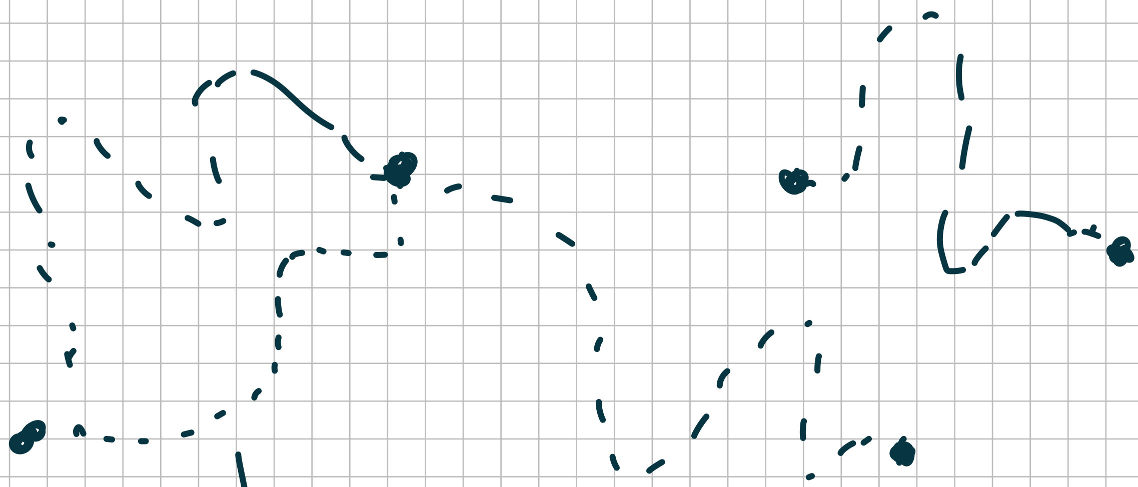
$\Updownarrow$

Galgp $\subset S_n$ is restricted.



Constraining alg. relations. $\Rightarrow$ small Gal. gp

— <u>contrast</u>.



non constr. relations (symmetric).

$\Rightarrow$ Big Gal gp.

Want to show if roots are expressible as nested radicals, then we are in the first situation.

In contrast, there are polys whose roots look like second sp.

Let K/F be a field ext in char 0. Say $\alpha \in K$ is any alg element.

$f(x) \in F[x]$ min. poly of α.

Thm: Let $G =$ Gal gp of $f(x)$.

If α is a nested radical then G is solvable.

Conversely if G is solvable then α is a nested radical.

S_2, S_3, S_4 & all their subgps are solvable.

But S_5 & A_5 are not!

G finite gp.

G is solvable if there is a chain of subgroups

$$G_0 = \{1\} \subset G_1 \subset G_2 \subset \cdots \subset G$$

such that $G_i \subset G_{i+1}$ is normal and G_{i+1}/G_i is cyclic of prime order.

Eqv.: ________ : cyclic
abelian

Ex. C_{18} is solvable

$$C_{18} = \mathbb{Z}/18$$

$$\underbrace{C_1 \subset C_3}_{C_3} \subset \underbrace{C_9}_{C_3} \underbrace{\subset C_{18}}_{C_2}$$

Any abelian gp is solvable.

$C_{18} \times C_9$

$S_2 = \mathbb{Z}/2\mathbb{Z}$ ✓ S_3 ✓

$$\underbrace{\{1\} \subset A_3}_{C_3} \underbrace{\subset S_3}_{C_2}$$

$$S_4 \underbrace{\supset}_{C_2} A_4 \underbrace{\supset}_{C_3} V = \{ id, (12)(34), (13)(24), (14)(32) \}$$

$$V \underbrace{\supset}_{C_2} \{ id, (12)(34) \} \underbrace{\supset}_{C_2} \{ id \}$$

"Solvable = Tower of cyclics of <u>prime order</u>"