From $\alpha \in \mathbb{C}$, we constructed $\mathbb{Q}[\alpha]$

$\mathbb{Q}[\alpha]$ = Smallest subring of $\mathbb{C}$ containing $\mathbb{Q}$ & α

$= \{ a_n \alpha^n + \cdots + a_1 \alpha^1 + a_0 \mid a_i \in \mathbb{Q} \}$

$\mathbb{Q}$-Vector space. $= \mathbb{Q}$-Span of $\{1, \alpha, \alpha^2, \ldots\}$

Inf. dim. $\mathbb{Q}$ vector space

Fin dim. $\mathbb{Q}$. vec. space.

$\alpha = \sqrt{2}$ $\quad \alpha^2 - 2 = 0$

$\alpha = \frac{1+\sqrt{5}}{2}$ $\quad \alpha^2 - \alpha - 1 = 0$

<u>Prop 1</u>: If α satisfies an equation of the form

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$$

where $a_i \in \mathbb{Q}$, then $\mathbb{Q}[\alpha]$ is finite dimensional.
Spanned by $1, \alpha, \ldots, \alpha^{n-1}$. so dim at most n.

Converse also holds.

Prop 2: If $\mathbb{Q}[\alpha]$ is finite dimensional, say $\dim = n$

then α must satisfy an equation of the form

$$\alpha^m + a_{m-1}\alpha^{m-1} + \cdots + a_0 = 0 \quad \text{for } m \leq n.$$

Proof: $1, \alpha, \alpha^2, \ldots, \alpha^n \in \mathbb{Q}[\alpha]$ are $(n+1)$ elts.

So these must be linearly dependant. $\therefore$

There exists $a_0, \ldots, a_n \in \mathbb{Q}$ so that (not all zero)

$$a_n\alpha^n + \cdots + a_0 = 0$$

In fact Prop 1 $\Rightarrow$ $m = n$.

$$\begin{tikzcd} & \mathbb{Q}[\alpha] \arrow[dl, "\text{inf dim}"'] \arrow[dr, "\text{fin dim}"] & \\ {} & & {} \end{tikzcd}$$

α does not satisfy any poly. eq^n with coeff in $\mathbb{Q}$.

dim $= n$ then α ~~satisfies~~ is a zero of a monic polynomial of degree n with $\mathbb{Q}$ coeff.

α is a root of a poly. with $\mathbb{Q}$ coeff.

α is **transcendental**

α is **algebraic**

Ex. π, e

$1, \sqrt{2}+\sqrt{3}, 5+2\sqrt{6}, 11\sqrt{2}+9\sqrt{3}, 49+20\sqrt{6}$

Ex. $\alpha = 3, \sqrt{3}, \sqrt{2}+\sqrt{3}$

$1, \sqrt{2}+\sqrt{3}, 5+\sqrt{6}\cdot 2, 5\sqrt{2}+5\sqrt{3}+4\sqrt{3}+6\sqrt{2}, 22+9\sqrt{6}+11\sqrt{6}+27, 11\sqrt{2}+9\sqrt{3},$

$1, \quad \sqrt{2}+\sqrt{3}, \quad 5+2\sqrt{6}$

$\shortparallel \qquad \shortparallel \qquad \shortparallel$

$\alpha^0 \qquad \alpha^1 \qquad \alpha^2$

$(X-5)^2 = 24$ sat. by $X=\alpha^2$

$(X^2-5)^2 = 24$ sat. by $X=\alpha$.

The set of algebraic numbers is countable.

$\mathbb{Q}[\alpha]$ $\qquad$ $\mathbb{R}[\alpha]$ $\qquad$ $\mathbb{C}[\alpha] = \mathbb{C}$.

$\uparrow \qquad \uparrow \qquad \uparrow$

$\mathbb{Q}$. v.space $\qquad$ $\mathbb{R}$. v.sp. $\qquad$ $\mathbb{C}$. v.sp.

$F[\alpha]$ $\qquad$ $F \subset \mathbb{C}$ subfield.

$\uparrow$

NOT. = inf dim. $\quad$ F v.sp. $\quad$ fin dim = α satis. poly eq^n with F-coeff

Transc. over F. $\qquad$ algebraic over F.