$\alpha \in \mathbb{C}$

Transcendental

- Does not ...

- $\mathbb{Q}[\alpha]$ is infinite dim $\mathbb{Q}$ vsp.

- eval at α : $\mathbb{Q}[x] \to \mathbb{Q}[\alpha]$ has 0 kernel.

Algebraic over $\mathbb{Q}$

- Satisfies a polynomial equation with $\mathbb{Q}$ coefficients

- $\mathbb{Q}[\alpha]$ is finite dim $\mathbb{Q}$ v.space

- eval at α : $\mathbb{Q}[x] \to \mathbb{Q}[\alpha]$ has non-zero ~~ideo~~ kernel.

$\mathbb{Q}[\alpha] \subset \mathbb{C}$

= All complex numbers that can be written as

$a_n\alpha^n + \cdots + a_1\alpha^1 + a_0$ for $a_i \in \mathbb{Q}$.

= ~~S~~ $\mathbb{Q}$-Span of $1, \alpha, \alpha^2, \ldots\ldots$

Consider x as a variable & look at $\mathbb{Q}[x]$

$\mathbb{Q}[x]$ = Poly. ring in x with coeff in $\mathbb{Q}$.

Have a ring homomorphism $\mathbb{Q}[x] \xrightarrow{e} \mathbb{C}$ eval. at $x = \alpha$

$$\begin{array}{ccc} \mathbb{Q}[x] & \xrightarrow{e} & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sum a_i x^i & \longmapsto & \sum a_i \alpha^i \\ p(x) & \longmapsto & p(\alpha) \end{array}$$

$\mathbb{Q}[\alpha]$ = Image of e.

$$\mathbb{Q}[x] \xrightarrow{e} \mathbb{C} \qquad \text{eval at } x=\alpha$$

Image of $e = \mathbb{Q}[\alpha]$.

$$\begin{array}{ccc} \mathbb{Q}[x] & \overset{e}{\twoheadrightarrow} & \mathbb{Q}[\alpha] \\ & & \cap \\ & & \mathbb{C} \end{array}$$

$I := \mathrm{Ker}(e) \subset \mathbb{Q}[x]$ an ideal.

$\quad\;\; \| $

$\{ p(x) \in \mathbb{Q}[x]$ that map to $0\}$

$I = (0)$	I is non-zero.
e is injective	e is not injective
α is transcendental.	α is algebraic

$$I \subset \mathbb{Q}[x] \xrightarrow{e} \mathbb{Q}[\alpha]$$

kernel.

The first iso thm. gives an isomorphism

$$\mathbb{Q}[x]/I \xrightarrow{\bar{e}} \mathbb{Q}[\alpha]$$

α transc. $\Rightarrow$ LHS $= \mathbb{Q}[x]$. so $\mathbb{Q}[x] \xrightarrow{\sim} \mathbb{Q}[\alpha]$

α algebraic : $I \neq 0$. $I \subset \mathbb{Q}[x]$.

All ideals of $\mathbb{Q}[x]$ are principal so

$$I = (p(x)).$$

means $p(\alpha) = 0$ & any $q(x)$ with $q(\alpha) = 0$ is of the form $q(x) = p(x) \cdot r(x)$

$p(x)$: The unique monic generator of I
is called the <u>minimal polynomial</u> of α.

$$(p(x)) = I \subset \mathbb{Q}[x] \twoheadrightarrow \mathbb{Q}[\alpha] \subset \mathbb{C}$$

$$\mathbb{Q}[x]/I \xrightarrow{\sim} \mathbb{Q}[\alpha] \longleftarrow \text{domain}$$

So $I = (p(x))$ is a prime ideal.
so $p(x)$ is an <u>irreducible</u> polynomial.
& up to scaling the only irred. poly in I.

$$\{ a + b\, 3^{1/5} \mid a, b \in \mathbb{Q} \} \subset \mathbb{C}.$$

✓ $\mathbb{Q}$-vsp.

✗ Not a subring.

$$3^{1/5} \cdot 3^{1/5} = 3^{2/5}$$

$\alpha = \sqrt{2}$ then $I = (x^2 - 2)$ (prove later)

$\alpha = -\sqrt{2}$ then $I = (x^2 - 2)$

$\alpha = 2^{1/3}$ $I = (x^3 - 2)$ $\leftarrow \mathbb{Q}[\alpha] \cong \mathbb{Q}[x]/(x^3 - 2)$

$\alpha = 2^{1/3} \cdot e^{i\pi/3}$ $I = (x^3 - 2)$