# Transcendental / algebraic

$F \subset K$ fields

$F \subset F[\alpha] \subset K$

- $\alpha$ does not ....
- $\alpha$ transc. over $F$
- $F[\alpha]$ inf dim $F$ vsp.
- $ev_\alpha$ has Kernel $0$

- $F[\alpha]$ not a field.

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- $\alpha$ sat. poly eq. coeff in $F$
- $\alpha$ algebraic over $F$
- $F[\alpha]$ fin. dim $F$ vsp
- $ev_\alpha \quad F[x] \longrightarrow F[\alpha]$ has kernel $= (p(x))$

  smallest deg sat. by $\alpha$

  min poly irreducible
- $F[\alpha]$ is a field.

Ex. $F = \mathbb{Q}$ $K = \mathbb{C}$ $\alpha = \sqrt[3]{2}$

$$F[\alpha] = \{ a \cdot 1 + b\, 2^{1/3} + c\, 2^{2/3} \mid a, b, c \in \mathbb{Q} \}$$

(min poly : $x^3 - 2$ $\leftarrow$ irreducible ? $\leftarrow$ defer.)

$F[\alpha]$ field means $\dfrac{1}{1 + 2^{1/3} - 2^{2/3}} \in F[\alpha]$

Prop: $\alpha$ alg over $F$ $\Rightarrow$ $F[\alpha]$ is a field.

Pf: $F[x] \xrightarrow{ev} F[\alpha]$ has $\ker = (p(x))$

$p(x)$ non-zero & irreducible.

$$F[x]/(p(x)) \xrightarrow{\sim} F[\alpha]$$

$(p(x))$ is a max ideal because $p(x)$ is irreducible.

$(p(x)) \subset I \subset F[x]$

$I = (q(x))$

$\Rightarrow q(x)$ divides $p(x)$

....

so $F[x]/(p(x))$ is a field.

Q: Why is $F[x]$ a PID?

Ex. $\mathbb{Q}[\sqrt{2}]$ field

$\mathbb{Q}[\sqrt{2}][\sqrt{3}] = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$

↳ smallest subring of $\mathbb{C}$ that contains $\mathbb{Q}$, $\sqrt{2}$, $\sqrt{3}$

$\mathbb{Q}[\sqrt{2}][2^{1/4}]$ → satisfies $\underbrace{x^2 - \sqrt{2}}$ over $\mathbb{Q}[\sqrt{2}]$

smaller in deg than $x^4 - 2$ over $\mathbb{Q}$

# Degree

$F \subset K \qquad \alpha \in K.$

degree of $\alpha$ over $F$ := deg of its min poly over $F$.

$\deg_{\mathbb{Q}}(\sqrt{2}) = 2 \qquad \deg_{\mathbb{Q}}(2^{1/3}) = 3$

$\deg_{\mathbb{Q}}(i) = 2 \qquad \deg_{\mathbb{R}}(i) = 2$

$\deg_F \alpha = \underbrace{\dim_F(F[\alpha])}_{\text{Generalises}}$

$F \subset A$

field subring of $A$ — ring

$\deg_F(A) := \dim A$ as $F$ v.sp.

$$\deg_{\mathbb{Q}}(\mathbb{Q}[\sqrt{2}]) = 2 \qquad \deg_{\mathbb{Q}}(\mathbb{Q}[\pi]) = \infty$$

$$\deg_{\mathbb{Q}}(\mathbb{Q}[\sqrt{2}, i]) = \ ?$$

$$\deg_{\mathbb{Q}[\sqrt{2}]}(\mathbb{Q}[\sqrt{2}, i]) = \deg_{\mathbb{Q}[\sqrt{2}]}(i) = 2$$

$$\underset{\mathbb{Q}[\sqrt{2}]}{\square} \cdot (1) + \underset{\mathbb{Q}[\sqrt{2}]}{\square} \cdot (i)$$

$$\deg_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] = 2 \qquad \square \longrightarrow \star \cdot (1) + \star \cdot (\sqrt{2})$$

$$\mathbb{Q}$$

$$\mathbb{Q}[\sqrt{2}, i] = \underbrace{\square}_{\mathbb{Q}[\sqrt{2}]} \cdot (1) + \underbrace{\square}_{\mathbb{Q}[\sqrt{2}]} \cdot (i)$$

$$= \boxed{\star \cdot 1 + \star \cdot \sqrt{2}} \cdot 1 + \boxed{\star \cdot 1 + \star \cdot \sqrt{2}} \cdot i$$

$$= \star\,(1 \cdot 1) + \star\,(\sqrt{2} \cdot 1) + \star\,(1 \cdot i) + \star\,(\sqrt{2} \cdot i)$$

<u>Thm</u>: $F \subset K \subset L$ fields

Then $\deg_F L = \deg_K L \cdot \deg_F K$