Degree

F a field, $F \subset K$, K also a field.

$\deg_F K = \deg(K/F) :=$ dim of K as an F v. space.

If finite, say that K is a finite ext^n of F.

Ex. α is an algebraic complex number then $\mathbb{Q}[\alpha]$ is a finite ext^n of $\mathbb{Q}$. & its deg = deg of min poly of α.

Notation:

$\mathbb{Q}(\alpha)$ = Smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ & α

α algebraic $\Rightarrow$ $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$

α transc.

$$\mathbb{Q}(\alpha) = \left\{ \frac{p(\alpha)}{q(\alpha)} \;\middle|\; p, q \in \mathbb{Q}[x] \right\}$$

Multiplicative property :

$$\underbrace{F \subset K}\underbrace{\subset L}$$ fields.

Thm : If K finite over F & L finite over K
then L finite over F &

$$\deg_F L = \deg_K L \cdot \deg_F K$$

Pf idea : Say $a_1, \dots, a_m$ is a basis of L over K.
Say $b_1, \dots, b_n$ is a basis of K over F.

Then $\{ b_i \cdot a_j \ , \ i = 1, \dots, m \ \ j = 1, \dots, m \}$ is a basis of L over F.

<u>Consequences</u>: $\mathbb{Q} \subset \mathbb{C}$

$\alpha, \beta \in \mathbb{C}$ algebraic over $\mathbb{Q}$.

Then $\alpha + \beta$ & $\alpha\beta$ & α/β are all alg. $/\mathbb{Q}$.

<u>Pf</u>: Consider

$$\underbrace{\underbrace{\mathbb{Q} \subset \mathbb{Q}[\alpha]}_{\text{finite}} \subset \mathbb{Q}[\alpha,\beta]}_{\text{finite}} \ni \begin{matrix} \alpha+\beta \\ \alpha\beta \\ \alpha/\beta \\ \alpha-\beta \end{matrix}$$

with $\mathbb{Q}[\alpha] \subset \mathbb{Q}[\alpha,\beta]$ finite.

($F \subset K$ finite & $\gamma \in K$ has to be alg $/F$.

$F \subset F[\gamma] \subset K$ so $F[\gamma]$ must be fin dim $/F$.)

If $F \subset K$ has degree d.

Then every $\alpha \in K$ has degree dividing d.

$$\underbrace{\underbrace{F \subset F[\alpha]} \subset K}.$$

Ex: $\sqrt[3]{2} \notin \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{17}]$

$$\downarrow$$

(#) $\Longrightarrow$ $X^3 - 2$ irred

will prove later

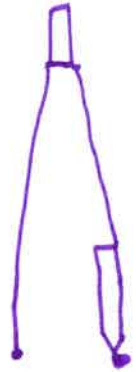
$$\Downarrow$$

deg 3.

$$\left.\begin{array}{l}
\mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{17}] \\
\quad \cup \quad \} \ 2 \text{ or } 1 \\
\mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}] \\
\quad \cup \quad \} \ 2 \text{ or } 1 \\
\mathbb{Q}[\sqrt{2}, \sqrt{3}] \\
\quad \cup \quad \} \ 2 \text{ or } 1 \\
\mathbb{Q}[\sqrt{2}] \\
\quad \cup \quad \} \ 2 \text{ or } 1 \\
\mathbb{Q}
\end{array}\right\} 2^{m}$$

Constructing using ruler & compass.

Ruler ↙ draws a line through 2 pts.



centre

Compass ↑ Draws a circle with given center & through a given pt.

Constructed ↙

Game: Start with some pts.

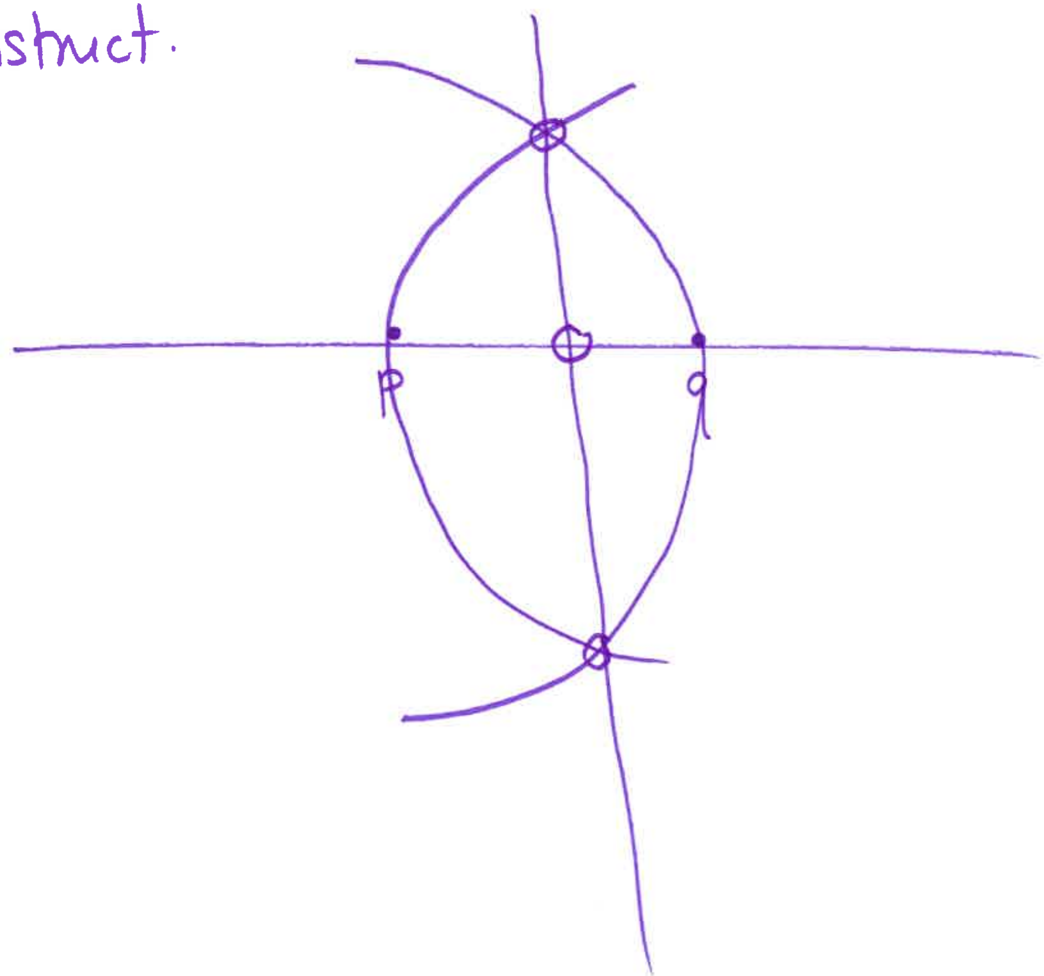
1. Draw a line through 2 constructed pts.
2. Circle centered at a cons. pt through a cons. pt.
3. Intersection pts are constructed.

<u>Problem</u> : Given

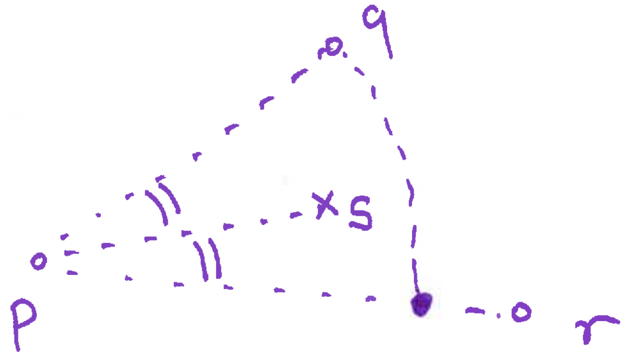
$\dot{p}$ $\dot{q}$.

<u>Task</u> : ~~Find~~ midpoint.

Construct.



Problem : Given

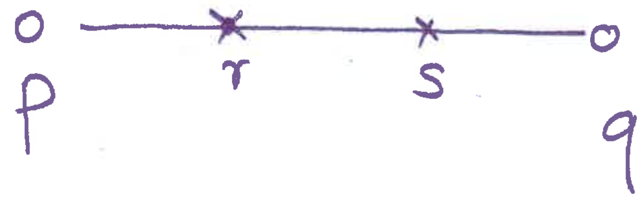


Want to construct a pt s so that $\measuredangle qpr$ is bisected by ps.

Problem: Trisect a segment.

Given

can be done.



Want r, s that trisect it.

Problem: Trisect an angle. ← cannot be done!