Ruler and compass constructions

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A fascinating application of the algebra we have done so far is the proof for the impossibility of certain geometric constructions. In particular, we will see that there does not exist a geometric construction using only ruler and compass that can trisect a given angle. But first, here are the rules of the game.

1. You are given a finite set of points in the plane. These points are assumed to be constructed.

2. The ruler allows you to draw a straight line between any two constructed points.

3. The compass allows you to draw a circle centered at a constructed point and passing through a constructed point.

You may add the intersection points of the lines and circles you draw to the set of constructed points, and use the ruler and compass any (finite) number of times.

Lots of cool things can be done.

1 Example

1. Given two points, construct their midpoint.
2. Given three points A, B, C, construct the angle bisector.
3. Given three points A, B, C, construct the unique circle passing through A, B, C.
4. Given two points A, B, divide the segment AB in 79 equal parts (or any other number).
5. ...

2 Algebraising the construction game

To bring algebra into the picture, we introduce coordinates. Let a set of constructed points S be given. Suppose $F \subset \mathbf{R}$ is a field that contains all the coordinates of S. The key idea is to explore in what ways F needs to be enlarged when we construct new points.

2.1 Using the ruler

Suppose we draw a line joining two points of S. Observe the following: we can write the equation of the line in the form

$$ax + by = c \tag{1}$$

where a, b, c are in the field F.

2.2 Using the compass

Suppose we draw a circle centered at a point of S passing through another point of S. Observe the following: we can write the equation of the circle in the form

$$(x - a)^2 + (y - b)^2 = c \tag{2}$$

where a, b, c are in the field F.

2.3 Taking intersections

Now is the most crucial part. We add new points by intersecting.

Intersecting two lines the intersection point of two lines of the form 1 has coordinates in F. So F need not be enlarged.

Intersecting a circle and a line the intersection point(s) of a line 1 and circle 2 have coordinates in $F[\sqrt{a}]$ for some $a \in F$.

Intersecting two circles the intersection point(s) of two circles 2 have coordinates in $F[\sqrt{a}]$ for some $a \in F$.

In summary, if we start with a set of constructed points whose coordinates lie in F, any new point we add must have its coordinates either in F or in $F[\sqrt{a}]$ for some $a \in F$. By repeating our reasoning, we get the following.

3 Proposition (Main)

Let P be a point constructed using the ruler and compass from a given set S. Assume that the coordinates of S lie in a field F. Then there exist extensions

$$F_0 = F \subset F_1 \subset \cdots \subset F_n$$

of the form $F_{i+1} = F_i[\sqrt{a_i}]$ for some $a_i \in F_i$ and such that the coordinates of P lie in F_n.

In particular, the degree of the extension of F generated by the coordinates of P is a power of 2.

4 Corollary

If the coordinates of P generate a transcendental extension of F or an extension whose degree is not a power of 2, then P cannot be constructed from S using ruler and compass.

5 Proposition

We start with the points $(0, 0)$ and $(0, 1)$. Then the point $(\cos 20, \sin 20)$ cannot be constructed. In particular, the 60-degree angle cannot be trisected, and hence there cannot exist a procedure that trisects a given angle.

5.0.1 Proof

We can take $F = \mathbf{Q}$. Trigonometry gives us the triple angle formula

$$\cos(3\theta) = 4\cos^3(t) - 3\cos(\theta).$$

So $\cos(20)$ satisfies the equation

$$1/2 = 4x^3 - 3x$$

or equivalently

$$8x^3 - 6x - 1 = 0.$$

This is irreducible mod 5 and hence irreducible. So $\cos(20)$ has degree 3 over $\mathbf{Q}$. But that means it is not constructible!