Factorisation in $\mathbb{Z}[x]$ vs $\mathbb{Q}[x]$

~~irred in $\mathbb{Z}[x]$~~ does not factor into poly. of lower deg.

R a ring $r \in R$ is reducible if it has a non-trivial factorisation.

$$r = s \cdot t \qquad s, t \in R \text{ neither a unit.}$$

Irreducible = Not ~~is~~ reducible.
= No non-trivial factorisation.

$3x^2 + 3 \in \mathbb{Z}[x]$ is reducible

$= 3 \cdot (x^2 + 1)$

$\in \mathbb{Q}[x]$ is irreducible.

Def: $f(x) \in \mathbb{Z}[x]$ is primitive if no prime p divides all the coeff.

Ex. $x^3 + 27$ primitive, $2x^2 + 4x + 3$ primitive

Thm: Let $f(x) \in \mathbb{Z}[x]$ be primitive polynomial.
Then $f(x)$ is reducible in $\mathbb{Z}[x]$ iff reducible in $\mathbb{Q}[x]$.

Obs: 1) Any $f(x) \in \mathbb{Z}[x]$ can be written as $d \cdot g(x)$ $\quad d \in \mathbb{Z}$, $g(x) \in \mathbb{Z}[x]$ primitive.

unique up to sign. = unique upto units of $\mathbb{Z}$.

2) Any $f(x) \in \mathbb{Q}[x]$ can be written as $f(x) = d \cdot g(x)$, $\quad d \in \mathbb{Q}$, $g(x) \in \mathbb{Z}[x]$ primitive.
unique upto sign.

$$\frac{1}{2}x^2 + \frac{2}{3}x = \frac{1}{6} \cdot (3x^2 + 4x)$$

$$= \frac{1}{12}(6x^2 + 8x)$$

$$= \frac{1}{12} \cdot 2 \cdot (3x^2 + 4)$$

Lemma (Gauss) :— Let $f(x), g(x) \in \mathbb{Z}[x]$ be primitive.

Then $f(x)g(x)$ is primitive.

Pf: ~~Take~~ Take a prime p. Look at

$$\mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]$$

$$f(x) \longmapsto \bar{f}(x) \quad \text{non-zero}$$

$$g(x) \longmapsto \bar{g}(x) \quad \text{non-zero}$$

$$f(x)g(x) \longmapsto \overline{f(x)} \cdot \bar{g}(x) \quad \text{non-zero}$$

$\square$.

Pf of Thm: $f(x)$ red. in $\mathbb{Z}[x] \Rightarrow f(x)$ reducible in $\mathbb{Q}[x]$.
(easy).

$f(x)$ red in $\mathbb{Q}[x]$. want $f(x)$ red. in $\mathbb{Z}[x]$.

$$f(x) = g(x) \cdot h(x) \quad , \quad g(x), h(x) \in \mathbb{Q}[x]$$

Write $g(x) = r_1 \cdot g_1(x) \quad r_1 \in \mathbb{Q} \quad g_1(x) \in \mathbb{Z}[x]$ prim.
$h(x) = r_2 \cdot h_1(x) \quad r_2 \in \mathbb{Q} \quad h_1(x) \in \mathbb{Z}[x]$ prim.

1. $\underbrace{f(x)}_{\text{prim.}} = (r_1 r_2) \underbrace{(g_1(x) \cdot h_1(x))}_{\text{prim.}}$ so $r_1 r_2 = \pm 1$

$$f(x) = \pm\, g_1(x) \cdot h_1(x) \quad , \quad h_1, g_1 \in \mathbb{Z}[x].$$

$\square$

$\mathbb{Z}[x]$ vs $\mathbb{Q}[x]$

$\mathbb{C}[t][x]$ vs $\mathbb{C}(t)[x]$

$F[t][x]$ vs $F(t)[x]$, F field.

$P_0(t) + P_1(t)x + \dots + P_n(t)x^n$

$P_i \in F[t]$ → primitive if no irred. poly in $F[t]$ divides all. $P_i(t)$.

~~For~~ prim. elt of $F[t][x]$, red. in $F[t][x]$ = red. in $F(t)[x]$.

$R[x]$ vs. (fr. field R) $[x]$ R has unique factorisation.

<u>Ex:</u> $x^3+x^2-t \in \mathbb{Q}(t)[x]$ is irreducible.

$x^3+x^2-t \in \mathbb{Q}[t][x]$ primitive.

$$\mathbb{Q}[t][x] = \mathbb{Q}[t,x] = \mathbb{Q}[x,t] = \mathbb{Q}[x][t]$$

$\cancel{t+x+t}$.

$x^3+x^2-t \in \mathbb{Q}[x][t]$ primitive

$x^3+x^2-t \in \mathbb{Q}(x)[t]$

$\hookrightarrow$ irred. (deg 1).