Adjoining roots: Fun. Thm. Alg.

$\mathbb{Q}$ want to solve $x^3 - 2 = 0$

can't, but can in $\mathbb{C} \supset \mathbb{Q}$

Can look at $\mathbb{Q}[2^{1/3}] \supset \mathbb{Q}$

$$x^5 - x^4 + 17x + 1 = 0$$

Thm (Fundamental Thm of algebra)

Given any $p(x) \in \mathbb{C}[x]$ of positive degree, there exists $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

$\mathbb{C}$ is "algebraically closed" field.

Def: A field F is **alg. closed** if any $p(x) \in F[x]$ of positive deg. has a zero in F.
non-constant.

Obs: 1) If F is alg. closed, then the only irred. poly in $F[x]$ are the linears.

2) If F is algebraically closed, then the only Finite ext^n of F is F itself.

$F \subset K$ finite $\Rightarrow$ $F = K$

Pf. $\alpha \rightsquigarrow$ min poly $p(x) \in F[x]$. must be linear.

so $\alpha \in F$.

3) $F \subset K$ alg. ext $\Rightarrow$ $F = K$.

$\mathbb{C} \subset \mathbb{C}(t)$

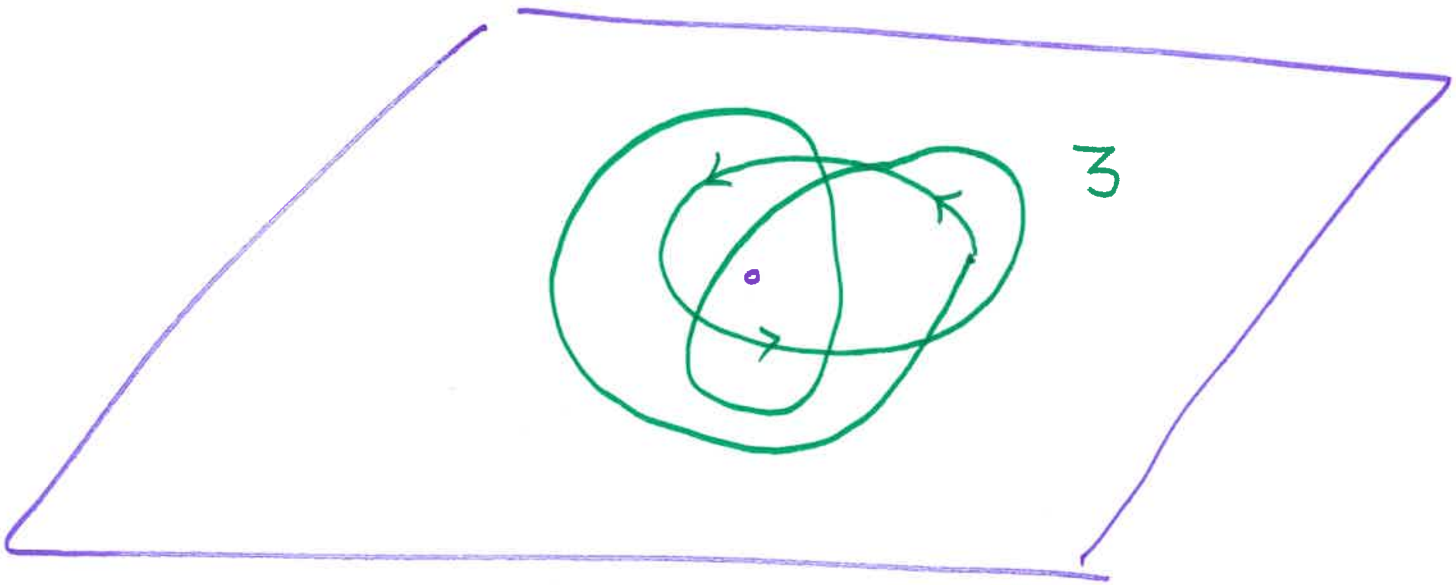
Not alg.

$\underline{\mathbb{C}}$ <u>is alg. closed</u>.

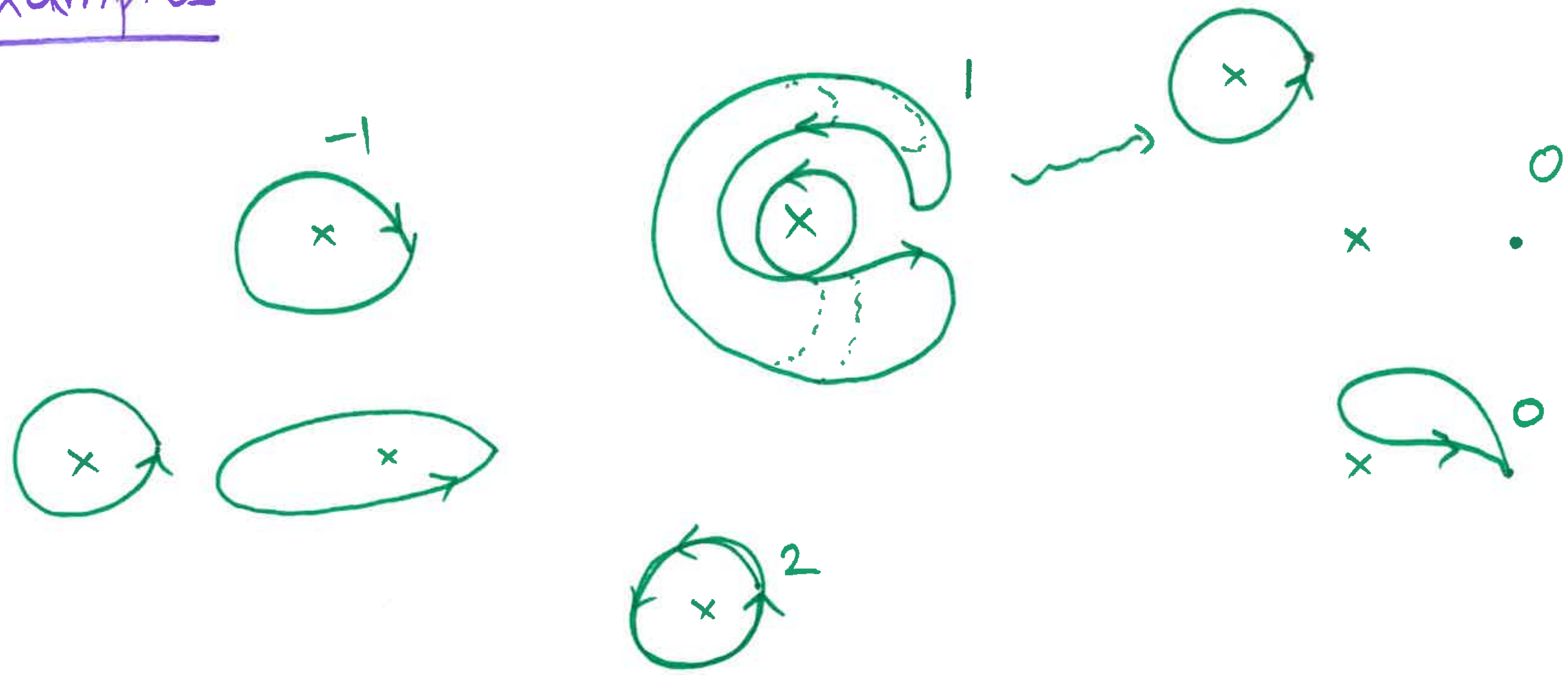
Start with $p(x) = x^n + \dots + a_0$, $a_i \in \mathbb{C}$

<u>Key input</u> (topology) :- "winding number".

Let γ be a closed curve in $\mathbb{C} \setminus \{0\}$



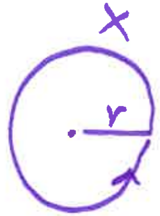
Examples:



Property: If γ is continuously deformed to γ', saying in $\mathbb{C} \setminus \{0\}$, ("homotopic") then γ & γ' have same winding number around 0.

Given: $p(x) = x^n + \cdots + a_0$ $\quad a_0 \in \mathbb{C}$.

Assume p has no zero.

Obs!: Consider $\{\cancel{p(x)}\}$ ~~as~~ $\gamma_r = \{ p(x) \mid$  $\}$

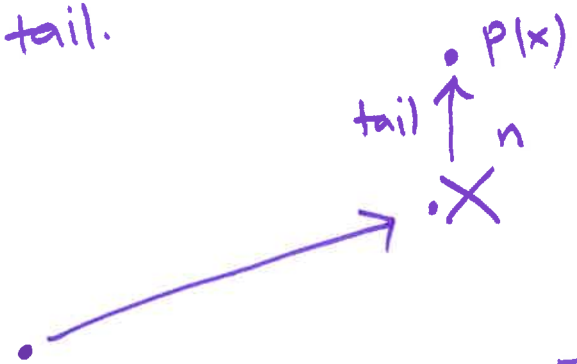
r large $\Rightarrow$ γ_r has winding number n.

Pf: $p(x) = \underbrace{x^n}_{\text{big}} + \underbrace{a_{n-1}x^{n-1} + \cdots + a_0}_{\text{small tail.}}$ $\qquad \underline{\underline{r}}$ big. $\quad r = |x|$

$\cancel{\gamma_{r,1}}$ $\quad \cancel{x^n} + t \cdot \cancel{\text{tail}}$

$\downarrow$

$\gamma_{r,0}$ $\quad \cancel{t=1 \to 0}.$



$\square$.

γ_r for r tiny.

a^* $p(x)$ a_0

$$p(x) = x^n + \dots + a_1 x + a_0$$

$$a_0 \neq 0$$

winding number of γ_r for r tiny $= 0$.

Varying r big to r tiny never hitting 0

$\Rightarrow$ $n = 0$.

$\square$