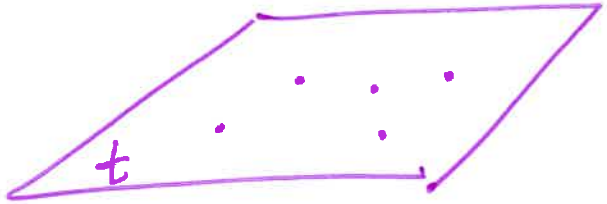
# Function fields

A finite extension of $\mathbb{C}(t)$.

$\mathbb{C}(t) \ni \dfrac{p(t)}{q(t)}$ :




$\longrightarrow \mathbb{C}$

$\mathbb{C}(t)[x] \big/ (x^2 - t)$

$x = $ "$\sqrt{t}$"
↳ A "multi-valued" "function" on the complex plane.

$\mathbb{C}(t)[x] \big/ (x^3 - 3tx^2 + 1)$

$x$ ~~roots of~~ $\cancel{x^3}$
↳ 3-valued "function" of $t$.

**Thm**: Let $\mathbb{C}(t) \subset K$ be a finite extⁿ.

Then $K \cong \mathbb{C}(t)[x] / f(x)$

where $f(x) \in \mathbb{C}(t)[x]$ irreducible.

**Thm** Let $F \subset K$ be a fin extⁿ of fields of char 0. Then $K \cong F[x] / f(x)$ for some irred $f(x)$.

(Primitive elt thm).

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$\mathbb{Q} \subset \mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2}+\sqrt{3}]$.

$$K = \mathbb{C}(t)[X] / (f(x))$$

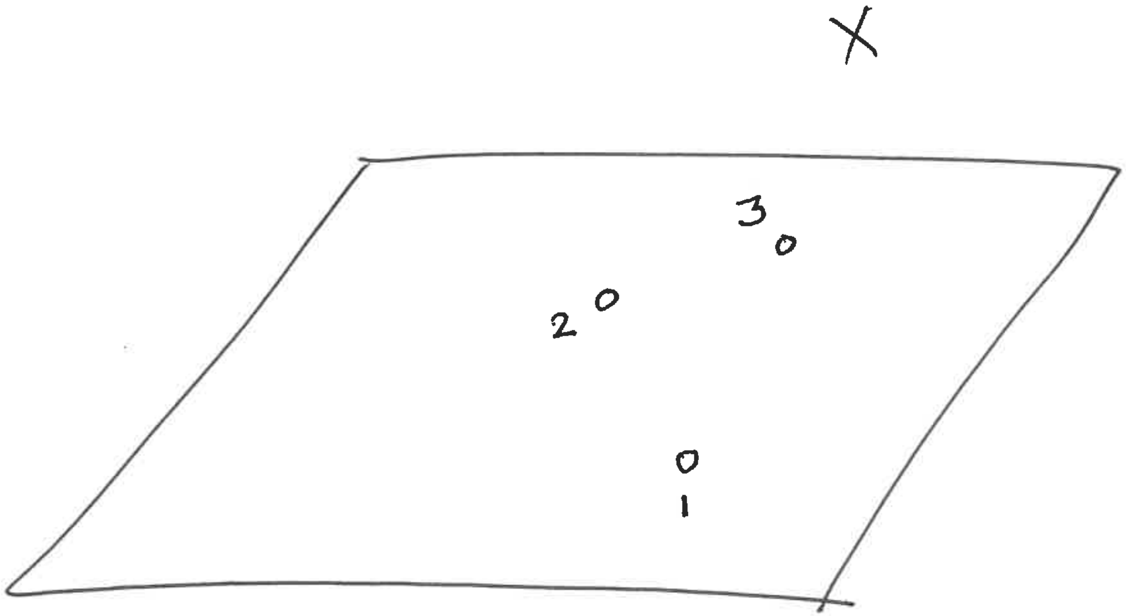
$$f(x) \in \mathbb{C}(t)[X]$$

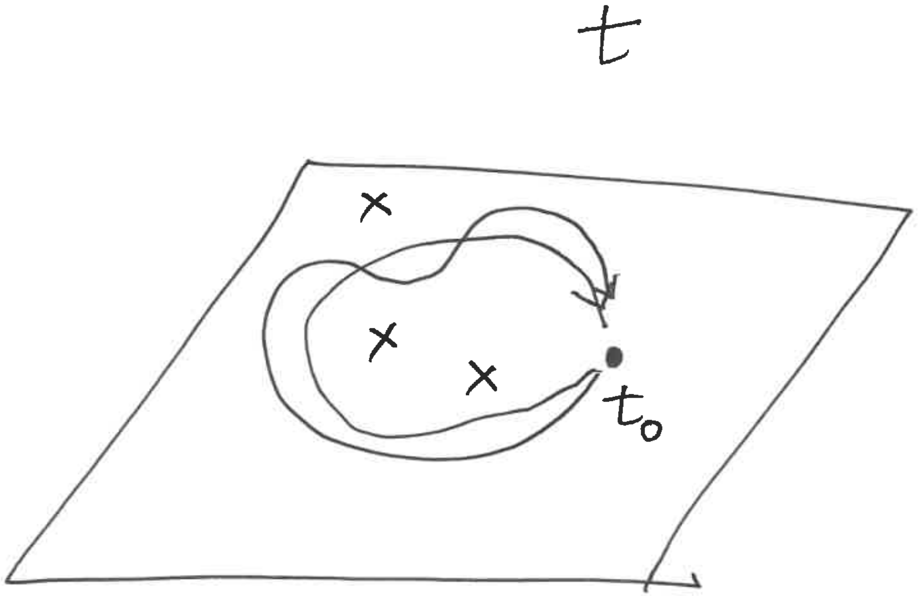
$$f(X) = a_n(t) \cdot X^n + a_{n-1}(t) X^{n-1} + \cdots + a_0(t)$$

By multiplying throughout, assume $a_i(t) \in \mathbb{C}[t]$

Then can think of $X$ as an $n$-valued function of $t$.

wiggle a path $\rightsquigarrow$ Same permutation.




o = Roots of $a_n(t)X^n + \cdots + a_0(t)$.

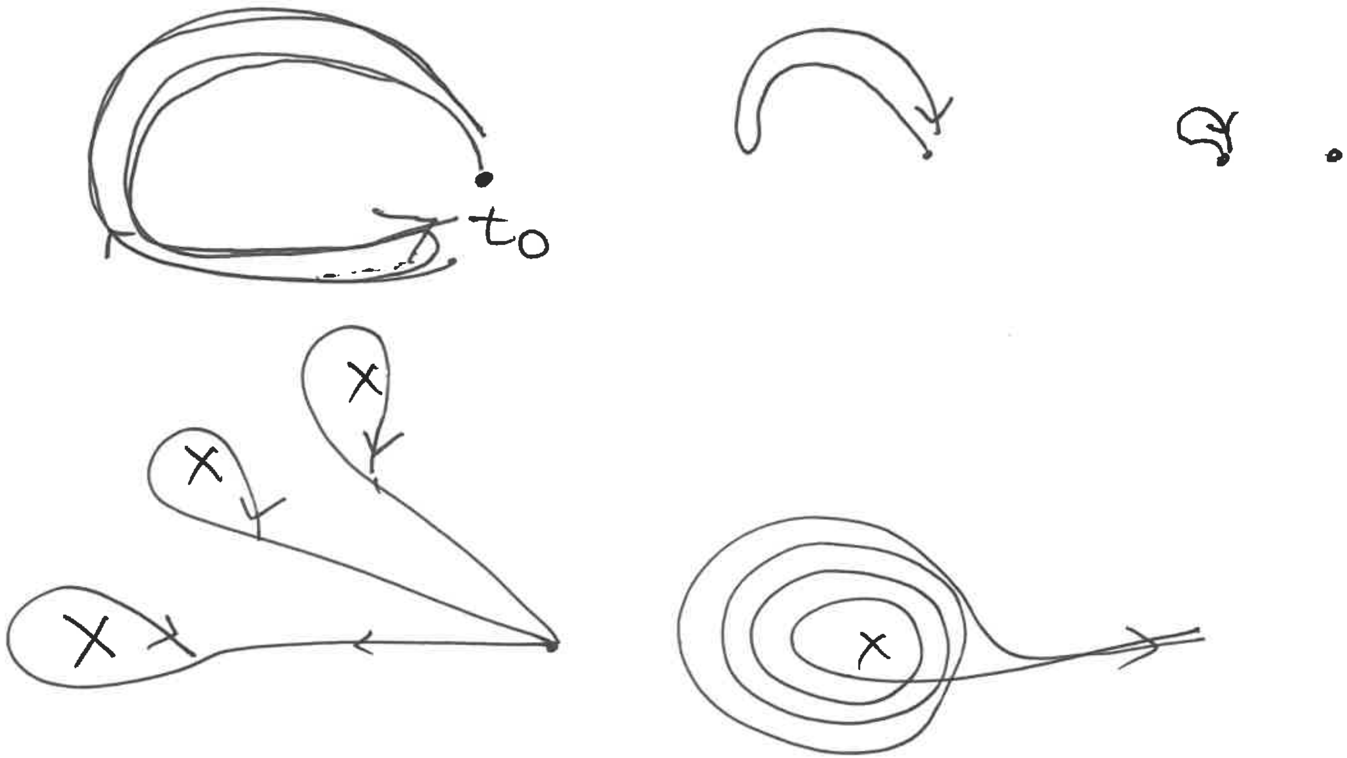
Each path based at $t_0$ $\rightsquigarrow$ Permutation of 1, 2, 3 ← roots. at $t_0$

reverse $\rightsquigarrow$ inverse permutation

Composition $\rightsquigarrow$ composition.

~~path~~

$\text{path}^{-1} \circ \text{path} \overset{\text{wiggle}}{\rightsquigarrow} \text{constant path}$

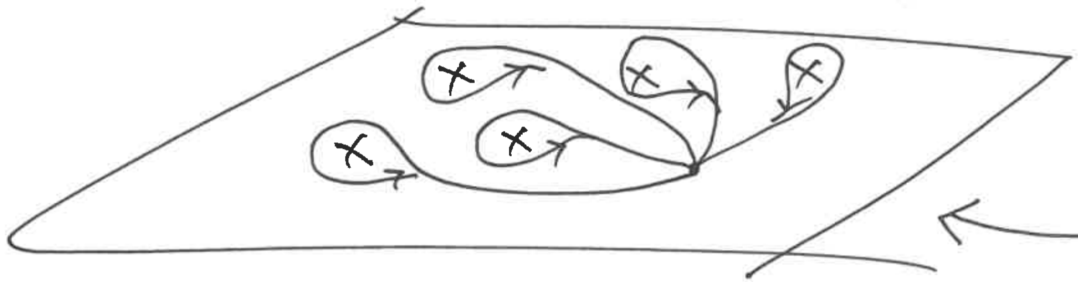


$U = \mathbb{C} \setminus \text{Bad pts}$ Fix $t_0 \in U$

$\pi_1(U, t_0) = \{\text{Paths based at } t_0\} / \text{wiggling}.$

$\hookrightarrow$ forms a group under composition of paths.

$\pi_1(U, t_0)$ is the free group gen. by lollipop paths.



Chasing roots gives a group hom

$$\pi_1(U, t_0) \longrightarrow S_n$$

"Monodromy homomorphism".

Field extⁿ of $\mathbb{C}(t)$ deg $n$ $\rightsquigarrow$ monodromy hom.