Galois Correspondence.

Given $F \subset K$ satisfying ... $\quad G = \mathrm{Aut}(K/F)$

Inter. Subfields $\longleftrightarrow$ Subgroups of G

$$\begin{array}{l} K \\ | \\ L \\ | \\ F \end{array} \Big\} \text{ degree} \quad \longrightarrow \quad \underbrace{\mathrm{Aut}(K/L) \subset G}_{\text{index.}}$$

Taking $L = K$ gives $\boxed{\underbrace{\deg(K/F)}_{\deg(L/F)} = |G|.}$

Today + tomorrow
Counting symmetries.

Counting

Setup: $F \subset K$ any finite extension

L any field, $F \to L$ ring hom.

$$\begin{tikzcd} K \arrow[r, dashed] & L \\ F \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} \mathbb{Q}[2^{1/4}] & \mathbb{C} \\ \mathbb{Q} \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd}$$

Q: How many $\dashrightarrow$ can there be?

Rules: Ring hom — ①

diagram commutes — ②

Example:

$$\begin{tikzcd} K = \mathbb{Q}[2^{1/4}] \arrow[r, dashed] & \mathbb{C} \\ F = \mathbb{Q}[\sqrt{2}] \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, "\sqrt{2} \mapsto \sqrt{2}"'] & \end{tikzcd}$$

Write a presentation of K over F

$$x \mapsto 2^{1/4}, -2^{1/4}$$

$$\begin{tikzcd} K \cong F[x] / (x^2 - \sqrt{2}) \arrow[r, dashed] & \mathbb{C} \\ F \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd}$$

$2^{1/4} \longleftrightarrow x$

$\Rightarrow$ 2 dotted arrows!

✓ $\Longleftarrow$

Replace $\mathbb{C}$ by $\mathbb{Q}[\sqrt{2}]$ $\Rightarrow$ 0 <u>dotted</u> maps.

$\dashrightarrow$ = Roots of $(x^2 - \sqrt{2})$ in the target field L.

$$
\begin{tikzcd}
\mathbb{Q}[\sqrt{2},\sqrt{3}] \arrow[rr, dashed] & & \mathbb{C} \\
& \mathbb{Q}[\sqrt{2}] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] \arrow[ul, phantom, "\cup"] & \\
\mathbb{Q} \arrow[uurr] \arrow[uu, phantom, "\bigcup"] \arrow[ur, phantom, "\subset"] & &
\end{tikzcd}
$$

2 dotted arrows.
sol'ns of (x^2-2) in $\mathbb{C}$.

For every intermediate $\dashrightarrow$, get two top $\dashrightarrow$.

So total 4 $\dashrightarrow$ = deg ext $\mathbb{Q} \subset \mathbb{Q}[\sqrt{2},\sqrt{3}]$

Prop:

$$\begin{tikzcd} K \arrow[r, dashed] & L \\ \cup & \\ F \arrow[uur] & \end{tikzcd}$$

Given. $F \subset K$ ext^n of deg n.

& $F \to L$.

Then # Ring homs $K \to L$ making the diagram commute is at most n.

Pf: **Case 1**: Suppose $K \cong F[x]/p(x)$

Then $\deg p(x) = n$.

Want.

$$\begin{tikzcd} F[x]/p(x) \arrow[r] & L \\ \cup & \\ F \arrow[uur, "\text{given.}"'] & \end{tikzcd}$$

$x \mapsto \alpha$

$\Rightarrow \alpha$ is a root of $p(x)$ in L.

$\Rightarrow$ at most n choices.

In general, induct on $n = \deg(K/F)$

$$\begin{tikzcd}
K \arrow[rr, dashed] \arrow[d, phantom, "\cup\ n/m"] \arrow[dd, dash, bend right, "n"'] & & L \\
F[\alpha] \arrow[urr, dashed] \arrow[d, phantom, "\cup\ m"] & & \\
F \arrow[uurr] & &
\end{tikzcd}$$

choose $\alpha \in K$
$\alpha \notin F$.

$m = \deg(\alpha/F)$.

How many intermediate $\dashrightarrow$?
At most m.

For every intermediate $\dashrightarrow$,
how many top $\dashrightarrow$? At most n/m
(ind. hyp.)

$\Rightarrow$ At most $m \cdot \left(\frac{n}{m}\right)$ top $\dashrightarrow$.

$\square$

Example:

$$\begin{tikzcd} K \arrow[r, dashed] & K \\ F \arrow[u, phantom, "\cup"] \end{tikzcd} \longrightarrow \mathrm{Aut}(K/F)$$

so $\left|\underline{\mathrm{Aut}(K/F)}\right| \leq \deg(K/F)$.

$\deg(K/F) \downarrow 4$

Can be strict

$F = \mathbb{Q}$ $\quad K = \mathbb{Q}[2^{1/4}]. \cong \mathbb{Q}[x]/(x^4-2)$

$$\begin{tikzcd} K \arrow[r, dashed] & L \\ \cup & \\ F \arrow[uur] & \end{tikzcd}$$

No. of $\dashrightarrow$ $\leq \deg(K/F)$.

<u>Q</u>: When can equality hold?

↳ L needs to be big enough to have sol^n to poly. we are solving.

- Poly we are solving must not have repeated roots.