Counting homs.

$$\begin{tikzcd} K \arrow[r, dashed] & L \\ \cup & \\ F \arrow[uur] & \end{tikzcd}$$

There at most $\deg(K/F)$ $\dashrightarrow$.

Ex.

$$\begin{tikzcd} \mathbb{Q}[2^{1/10}] \arrow[r, dashed] & \mathbb{R} \\ \cup & \\ \mathbb{Q}[\sqrt{2}] \arrow[uur, "\sqrt{2} \mapsto -\sqrt{2}"'] & \end{tikzcd}$$

$\mathbb{Q}[\sqrt{2}][x] / (x^5 - \sqrt{2}) \dashrightarrow \mathbb{R}$

$x \mapsto$ Roots in $\mathbb{R}$ of $x^5 + \sqrt{2}$

(only one).

Q: When do we have the max number of $\dashrightarrow$?

$$\begin{array}{ccc} K & & \\ \cup & & L \\ F & \nearrow & \end{array}$$

Two obstacles —

① Poly. don't factor enough in $L[x]$

② Poly. may have repeated roots.

Prop: Given

$$\begin{tikzcd} K \arrow[r, dashed] & L \\ \cup & \\ F \arrow[uur] & \end{tikzcd}$$

Assume: for every $\alpha \in K$, the min. poly of α in $F[x]$ factors into distinct linear factors in $L[x]$. (*)

Then there are $\deg(K/F)$ homs $K \to L$, making the triangle commute.

Pf: Case 1: $K \cong F[x]/p(x)$. $\quad n = \deg(K/F) = \deg p(x)$

Want
$$\begin{tikzcd} K \arrow[r, dashed] & L \\ F \arrow[u, phantom, "\cup" rotate=-90] \arrow[ur] & \end{tikzcd}$$
$x \longmapsto$ Root of $p(x)$ in L.

By ⊛, $p(x)$ has n distinct roots in L. $\Rightarrow$ n homs.

In general
$$\begin{tikzcd} K & & L \\ & F[\alpha] \arrow[ul, phantom, "\cup\ n/m"] \arrow[ur, dashed] & \\ F \arrow[uu, phantom, "n\ \cup"] \arrow[ur, phantom, "\cup_m"] \arrow[uurr] & & \end{tikzcd}$$

m intermediate arrows.

Fix an arrow $\dashrightarrow$.

Make sure ⊛ holds for

$$\begin{tikzcd} \beta \in K & L. \\ F[\alpha] \arrow[u, phantom, "\cup" rotate=-90] \arrow[ur] & \end{tikzcd}$$

$q(x) = $ min. poly of β in $F[\alpha][x]$. $\quad q(x)$ divides $r(x)$ in $F[\alpha][x]$

$r(x) = $ min poly of β in $F[x]$

Know that $r(x)$ splits into dist. lin. facts. in $L[x]$

$\Rightarrow$ $q(x)$ also does. $\Rightarrow$ ⊛ continues to hold.

$\square$.

(*) : No repeated roots.
Existence of roots.

Prop: Suppose F has char 0. Let $p(x) \in F[x]$ be irred. Then $p(x)$ cannot repeated roots in any ext^n of F.

Pf: Consider $p'(x) \leftarrow$ non-zero polynomial.

derivatives of non-const poly are non-zero in char 0.

$\gcd(p(x), p'(x)) = 1$ because $p(x)$ is irred & $p(x)$ can't divide $p'(x)$.

$\Rightarrow \exists\ a(x), b(x) \in F[x]$ s.t.

$1 = a(x)\,p(x) + b(x)\,p'(x)$. holds in $F[x]$ also in $K[x]$ for $K \supset F$.

$\Rightarrow$ $p(x)$ & $p'(x)$ remain rel. prime in $K[x]$. $\Rightarrow$ No repeated root in K.

$$\begin{tikzcd}
K & L \\
U & \\
F \arrow[uur] &
\end{tikzcd}$$

(*) $\forall \alpha \in K$
min poly of α in $F[x]$
has distinct lin factors in $L[x]$.

In char 0. distinct is automatic.

Only question: Do~~es~~ we have all lin. factors?

YES if L is alg. closed ✓

YES:

$$\begin{tikzcd}
K & K \\
U & \\
F \arrow[uur, bend right] &
\end{tikzcd}$$

$F \subset K$ splitting field.

Concl: If $F \subset K$ is a splitting field in Char 0 then

$$\begin{array}{ccc} K & \dashrightarrow & K \\ \cup & & \cup \\ & F & \end{array}$$

there are exactly $\deg(K/F)$ $\dashrightarrow$.

$$\Rightarrow \quad |\mathrm{Aut}(K/F)| = \deg(K/F).$$