

Galois Correspondence

$F \subset K$ Galois extension. finite. $G = \text{Aut}(K/F)$



Yesterday : $\xrightarrow{\hspace{2cm}}$ top followed by bottom $\xleftarrow{\hspace{2cm}} = \text{id.}$

\Rightarrow Finitely many int. fields.

\Rightarrow If F is infinite, then the union of strict subfields of $K \subsetneq K$.

Primitive elt thm. $\Rightarrow \exists \alpha \in K$ st. $F(\alpha) = K$.
 ↳ Pick α to not be in any intermediate subfield.

what about non Galois ext?

Assume $\text{char } \mathbb{Q} = 0$.

Then every finite ext FCK is contained in a finite Galois extn. $\exists L/FCKCL$ s.t. FCL is Galois finite.

then FCK also has fin. many intermed. fields.

so avoid their union. $\Rightarrow \exists \alpha \in K$ s.t. $K = F(\alpha)$.

primitive elt thm.

Ex.

$$\mathbb{Q} \subset \mathbb{Q}[2^{\frac{1}{3}}, 5^{\frac{1}{4}}] \subset \mathbb{Q}[2^{\frac{1}{3}}, \cancel{\zeta_3}, 5^{\frac{1}{4}}, i]$$

||
Splitting field of
 $(x^3 - 2) \cdot (x^4 - 5)$.
over \mathbb{Q} .

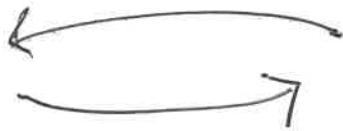
In general ($\text{char } F = 0$)

$$F \subset K = F[\alpha_1, \alpha_2, \dots, \alpha_n] \subset L$$

$P_i(x) = \min \text{ poly of } \alpha_i \in F[x]$
L is a splitting field of $\underbrace{P_1(x) \cdots P_n(x)}_{F[x]}$ over K.

Then L is splitting field of $\prod P_i(x)$ over F.

Fields



Groups.

Last thing:



followed by



= id.

$$K^H \subset K$$

$$H \subset G$$

Let's show

$$\frac{\deg(K/K^H)}{|\text{Aut}(K/K^H)|} = |H|. \text{ Then we are done.}$$

(because

$$\text{Aut}(K/K^H) \supset H \quad \& \quad |\text{Aut}(K/K^H)| \\ = \deg(K/K^H).$$

Enough:

$$\deg(K/K^H) \leq |H|.$$

Know

$$K = \bigcup_{\alpha \in H} K^H(\alpha) \quad \text{for some } \alpha \in K.$$

$$(x-\alpha)(x-h_1\alpha)(x-h_2\alpha)\dots = \prod_{h \in H} (x-h\alpha)$$

$$P(x) = \prod_{h \in H} (x - h\alpha)$$

coeff in K^H ?

$$\text{Const. term} = \prod_{h \in H} (h\alpha)$$

$$h(h_1(\alpha)) = (\underline{h \cdot h_1})(\alpha)$$

Applying an elt of H just permutes $\{\alpha, h(\alpha), h_2\alpha, \dots\}$

All coeff are symmetric in the roots. \Rightarrow Unchanged by permutations.

so $P(x) \in K^H[x]$.

□.

K
 L
 F

$\deg m_i = \underline{\text{size}}$
 index



$|\text{Aut}(K/F)|$
 \cup
 $\boxed{\text{Aut}(K/L)}$
 $\leftarrow \text{size } m$