Galois Correspondence

$F \subset K$ Galois extension. finite. $G = Aut(K/F)$

Intermediate fields $\underset{\longleftarrow}{\longrightarrow}$ Subgps of G

Yesterday : $\xrightarrow{}$ top followed by bottom $\xleftarrow{}$ = id.

$\Rightarrow$ Finitely many int. fields.

$\Rightarrow$ If F is infinite, then the union of strict subfields of K $\subsetneq K$.

Primitive elt thm. | $\Rightarrow$ $\exists$ $\alpha \in K$ s.t. $F(\alpha) = K$.

$\hookrightarrow$ Pick α to not be in any intermediate subfield.

$\hookrightarrow$ what about non Galois extⁿ?

Assume char 0.

Then every finite ext $F \subset K$ is contained in a finite Galois ~~ext~~ ext^n: $\exists$ L $F \subset K \subset L$ s.t. $F \subset L$ is Galois. finite.

then $F \subset K$ also has fin. many intermed. fields.

So avoid their union. $\Rightarrow$ $\exists$ $\alpha \in K$ s.t. $K = F(\alpha)$.

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primitive elt thm.

Ex. $\mathbb{Q} \subset \mathbb{Q}[2^{1/3}, 5^{1/4}] \subset \mathbb{Q}[2^{1/3}, \zeta_3, 5^{1/4}, i]$

$\qquad\qquad\qquad\qquad ||$

Splitting field of $(x^3-2)\cdot(x^4-5)$ over $\mathbb{Q}$.

In general (char $F = 0$)

$$F \subset K = F[\alpha_1, \alpha_2, \ldots, \alpha_n] \subset L$$

$P_i(x)$ = min poly of α_i $\in F[x]$

L is a splitting field of $\underbrace{P_1(x) \cdots P_n(x)}_{F[x]}$ over K.

Then L is splitting field of —〃— over F.

Fields $\rightleftarrows$ Groups.

Last thing: $\leftarrow$ followed by $\rightarrow$ $=$ id.

$$K^H \subset K \longleftarrow H \subset G$$

Let's show $\underline{\underline{\deg(K/K^H) = |H|}}$. Then we are done.

(because $\mathrm{Aut}(K/K^H) \supset H$ & $|\mathrm{Aut}(K/K^H)| = \deg(K/K^H)$.

Enough: $\deg(K/K^H) \leq |H|$.

Know $K = \underline{\underline{K}}^H(\alpha)$ for some $\alpha \in K$.

$$(X-\alpha)(X-h_1\alpha)(X-h_2\alpha)\cdots = \prod_{h \in H}(X - h\alpha)$$

$$p(x) = \prod_{h \in H} (x - h\alpha)$$

coeff in K^H ?

$h(h_1(\alpha)) = (h \cdot h_1)(\alpha)$

Const. term $= \prod_{h \in H} (h\alpha)$

Applying an elt of H just permutes $\{\alpha, h(\alpha), h_2\alpha, \ldots\}$

All coeff are symmetric in the roots. $\Rightarrow$ unchanged by permutations.

so $p(x) \in K^H[x]$.

$\square$.

$$\begin{array}{ccc} K & & \\ \cup & \Big\}\ \text{deg } m. = \underline{\text{size}}. & \\ & & L \\ & \Big\}\ \underline{\text{index}}. & \\ F & & \end{array} \qquad \longleftrightarrow \qquad \begin{array}{c} |\mathrm{Aut}(K/F)| \\ \cup \\ \boxed{\mathrm{Aut}(K/L)} \end{array} \leftarrow \text{size } m$$