Solving equations

$x^2 + ax + b = 0$ $\qquad$ $x = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

Char 0 for simplicity.

cubic: $\quad$ Ex. $\quad x^3 + 3x + 1$ $\qquad$ $x^3 + bx^2 + cx + d = 0$

Idea:

Field gen. by roots $\quad (\mathbb{Q}[\alpha, \beta, \gamma] \subset \mathbb{C})$.

Find intermediate fields. $\quad \cup \qquad \cup$

Base field = Field of coeff. $\quad (\mathbb{Q})$

$\mathbb{Q} \subset \mathbb{Q}[\alpha,\beta,\gamma]$ $\quad$ α,β,γ roots of a given cubic

$x^3+bx^2+cx+d=0$ $\quad$ (e.g. x^3+3x+1)

irreducible. $\quad$ (x^3+3x+2)

$G = \mathrm{Aut}(\mathbb{Q}[\alpha,\beta,\gamma]/\mathbb{Q})$.

$G \hookrightarrow S_3 =$ Permutations of α,β,γ.

Subgroups of S_3: $\{1\}$, S_3, $\langle(12)\rangle$, $\langle(23)\rangle$, $\langle(13)\rangle$, $\langle(123)\rangle$

irreducibility rules out $\underline{\underline{\{1\}}}$, $\underline{\underline{\langle(12)\rangle}}$, $\underline{\underline{\langle(23)\rangle}}$, $\underline{\underline{\langle(13)\rangle}}$.

$\hookrightarrow$ explore this for higher degrees.

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Quartic $\quad$ $\mathbb{Q} \subset \mathbb{Q}[\alpha,\beta,\gamma,\delta]$ $\quad$ roots of irred. quartic.

$G = \langle (\alpha\beta)(\gamma,\delta), (\alpha\beta) \rangle$ $\quad$ $\underbrace{(x-\alpha)(x-\beta)}_{\text{base field}} \cdot \underbrace{(x-\gamma)(x-\delta)}_{\text{base field.}}$

Contradicts irreducibility.

G cannot fix a $\overset{\text{strict}}{\text{subset}}$ of the roots.

All roots must form one orbit under G.

Prop: $F \subset K$ finite Galois ext^n. $G =$ Gal group.

~~$\exists$~~ $\forall\ \alpha \in K$ the roots of the min. poly of α form one orbit under G.

This can be used to find min poly's.

Given $\alpha \in K$ what's the min poly?

Let $\alpha_1, \ldots, \alpha_m$ be the G-orbit of α.

Then min poly of α $= (X-\alpha_1)\cdots(X-\alpha_m)$.

So $\deg \alpha / F = \#$ Orbit of α.

C_3 or S_3 : Which one? (Why does it matter?).

$$\begin{array}{ccl} & \mathbb{Q}[\alpha,\beta,\gamma] & \\ & \cup & \Big\} \text{ cubic.} \\ \mathbb{Q}[\delta] = & \mathbb{Q}[\alpha,\beta,\gamma]^{C_3} & \\ & \cup & \Big\} \text{ quadratic} \\ & \mathbb{Q} & \end{array}$$

Δ = Discriminant.

$$\delta = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$$ has orbit $\{\delta, -\delta\}$

$$= \sqrt{(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2}$$

$$= \sqrt{\Delta}$$

Δ is expressible in terms of the coeffs!

Wikipedia.