Disc of $x^3 + px + q$ is

$$\Delta = -4p^3 - 27q^2$$

$x^3 + ax^2 + \dots$

$(x-\frac{a}{3})^3 + a(x-\frac{a}{3})^2 + \dots$

<u>Ex.</u> $x^3 + 3x + 2$

$\Delta = -4(27) - 27(4)$
$= -216$

$\cancel{x^3+3x+1}$ $x^3 - 3x + 1$

$\Delta = +4(27) - 27$
$= 81$

$\Delta = \delta^2$ $\delta = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$ $\delta \in \mathbb{Q}$.

If $\delta \in \mathbb{Q} \Rightarrow$ Galois group does not have odd permutations.

Let $F \subset K$ be a Galois extension, splitting field of $p(x) \in F[x]$ ~~with~~ $\cancel{p(x)}$ such that $p(x)$ has n distinct roots $\alpha_1, \ldots, \alpha_n \in K$.

$\boxed{\text{char } 0}$

$G = \mathrm{Aut}(K/F) \hookrightarrow S_n$

Consider $\delta = \prod_{i>j} (\alpha_i - \alpha_j) \in K$, $\Delta = \delta^2 \in F$.

If Δ is a square in F, then $G \subset A_n = \{\text{even permutations}\}$

Conversely, if $G \subset A_n$, then Δ is a square in F.

$p(x)$ irred cubic. K/F splitting field

What is $G = \mathrm{Aut}(K/F)$.

S_3 iff Δ is not a square.

$x^3 - 3x + 2$.

A_3 iff Δ is a square

e.g. $x^3 - 3x + 1$

F & cubic $p(x) = X^3 + pX + q$

$$\begin{tikzcd}
K \arrow[dd, dash] \arrow[dr, dash, "\text{deg } 3."] & \\
 & F[\sqrt{\Delta}] \\
F \arrow[ur, "\checkmark", "\text{deg } 2 \text{ or } 1"'] &
\end{tikzcd}$$

Kummer to the rescue. (char 0)

Let $F \subset K$ be a Galois ext^n with Gal. gp $\mathbb{Z}/p\mathbb{Z}$.

Assume F has all p^{th} roots of 1. p prime.

(i.e. $X^p - 1$ splits completely in F).

Then: K is obtained from F by adjoining a p^{th} root.

That is, $\exists \; a \in K$ s.t. $b = a^p \in F$ and $a \notin F$.

(Then $K \cong F[x] / \underset{(X^p - b)}{(X^p - a^p)}$).

Ex. $p(x) = x^3 - 3x + 1$ $\qquad F = \mathbb{Q}[\zeta_3]$

$\underbrace{F \subset K}_{\mathbb{Z}/3\mathbb{Z}} =$ splitting field of $p(x)$ over F. $\subset \mathbb{C}$

Kummer: $K = F[a]$ for some a whose cube $\in F$

$K = F[\sqrt[3]{b}]$ for some $b \in F$.

In our case $K = F[\alpha, \beta, \gamma]$ $\quad \alpha, \beta, \gamma$ are the roots.

$a = \alpha + \zeta_3 \beta + \zeta_3^2 \gamma$ $\qquad$ Claim: $a^3 \in F$.