Finishing the cubic

Given $p(x)$ a cubic $\quad F = \mathbb{Q}[\zeta_3]$

$P(x) = x^3 + px + q$.

$$\begin{array}{c} K = F[\alpha, \beta, \gamma] \\ | \\ F[\sqrt{\Delta}] \\ | \\ F \end{array}$$

$\Delta = -4p^3 - 27q^2$

① — $u = \alpha + \zeta_3 \beta + \zeta_3^2 \gamma \Rightarrow u^3 \in F[\sqrt{\Delta}]$

↳ An exp in ~~coeff~~ F & $\sqrt{\Delta} = \delta$ → can find.

$u = \sqrt[3]{\text{Exp in coeff } \& \sqrt{\Delta}}$

② — $v = \alpha + \zeta_3^2 \beta + \zeta_3 \gamma \quad v^3 \in F[\sqrt{\Delta}] \quad v = \sqrt[3]{\text{Exp in coeff } \& \sqrt{\Delta}}$

③ — $0 = \alpha + \beta + \gamma$

Solve 3 linear eq^{n}s ①, ②, ③ get α, β, γ.

Lesson : Suppose have $p(x) \in F[x]$ &

F has enough roots of 1. Let K be splitting field of $p(x)$.

$p_1, \dots, p_\ell$

All $p \mid n!$ ✓

$$F = F_0 \subset F_1 \subset \cdots\cdots \subset F_\ell = K$$

S.t. $F_i \subset F_{i+1}$ is Galois of degree p_i ← prime.

Then the roots of $p(x)$ can be written in terms of elts of F & $\sqrt[p_i]{}$ symbols.

$\sqrt[p_i]{}$

$$\underbrace{F \qquad \sqrt[p_2]{\underbrace{F + \sqrt[p_1]{\underline{\quad F \quad}}}}}$$

$\pm\sqrt{}$

Task: ① Translate $F_0 \subset \underbrace{F_1 \subset}_{\text{Galois of order } p_i} \cdots \subset F_\ell = K$

into a ladder for $G = \mathrm{Gal}(K/F_0)$

② Prove Kummer's thm.

$F \subset K$ ~~cyclic~~ Galois with $\mathrm{Aut}(K/F) \cong \mathbb{Z}/p\mathbb{Z}$.

Want an $\alpha \in K$ $\alpha \notin F$ s.t. $\alpha^p \in F$.

Let σ be a gen. of $\mathrm{Aut}(K/F)$.

~~Suppose~~ Let us find $\alpha \in K$ s.t. $\sigma(\alpha) = \zeta_p^i \cdot \alpha$

$i = 1, \dots, p-1$

$\sigma : K \to K$ has eigenvalue ζ_p^i for some $i = 1, \dots, p-1$.