Solvable Groups

G solvable if $\exists$ chain

$$G \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n \rhd \{1\}$$

with $G_{i+1}/G_i \cong \mathbb{Z}/p_i\mathbb{Z}$

Ex: 1) S_4 0) $\{1\}$ 2) $\mathbb{Z}/p_i\mathbb{Z}$ 3) S_3

4) Any finite abelian group.

$$\mathbb{Z}/4 \times \underbrace{\mathbb{Z}/9 \supset \mathbb{Z}/2}_{\mathbb{Z}/2} \times \underbrace{\mathbb{Z}/9 \supset 1 \times \mathbb{Z}/9}_{\mathbb{Z}/2} \underbrace{\supset 1 \times \mathbb{Z}/3}_{\mathbb{Z}/3} \underbrace{\supset \{1\}}_{\mathbb{Z}/3}.$$

5) $U_2(\mathbb{F}_3)$

$$= \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \text{ entries in } \mathbb{F}_3 \right\} = U$$

$$U = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \xrightarrow{\text{diag.}} \mathbb{F}_3^\times \times \mathbb{F}_3^\times$$

$$\underset{\underline{\text{Ker.}}}{U_1} \hookrightarrow U \xrightarrow[\text{diag}]{\text{lower}} \mathbb{F}_3^\times \qquad \underset{\text{ker}}{U_2} \hookrightarrow U_1 = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\} \xrightarrow[\text{diag}]{\text{upper}} \mathbb{F}_3^\times$$

$$\underbrace{U \rhd U_1}_{C_2} \underbrace{\rhd U_2}_{C_2} \underbrace{\rhd \{1\}}_{C_3}. \qquad U_2 = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \cong \mathbb{F}_3$$

<u>Thm</u>: Let G be a finite solvable gp.

① Any subgp of G is solvable.

② Any quotient of G is solvable.

~~③ If~~

(intersect series for G with a sub or map series for G onto the quotient.)

<u>Thm</u>: G fin. gp. $N \lhd G$ normal. $H = G/N$. If N & H are solvable then G is solvable.

$$\mathbb{F}_3 \cong U_2 \lhd U \to \mathbb{F}_3^\times \times \mathbb{F}_3^\times$$

Pf: N solvable & $H = G/N$ solvable.

$$\underbrace{G \triangleright N}_{H} \triangleright \overbrace{N_1 \triangleright N_2 \triangleright \cdots \triangleright \{1\}}$$

$\mathbb{Z}/p_i$ quotients.

$$G_i = \{ g \in G \mid \bar{g} \in H_i \}$$

$$\begin{tikzcd}
\overbrace{G \arrow[d] \triangleright G_1}^{\mathbb{Z}/p_i} \arrow[d] \triangleright \overbrace{G_2}^{} \arrow[d] & \cdots & \triangleright N \\
\underbrace{H \triangleright H_1}_{\mathbb{Z}/p_i} \triangleright \underbrace{H_2}_{} \triangleright & \cdots & \triangleright \{1\}
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \triangleleft G \arrow[r] \arrow[dr, "\ker \to"'] & H \arrow[d] \\
& H/H_1
\end{tikzcd} \qquad g \mapsto \bar{g}$$

$$\begin{tikzcd}
G_2 \triangleleft G_1 \arrow[r] \arrow[dr, "\ker \to"'] & H_1 \arrow[d] \\
& H_1/H_2
\end{tikzcd}$$

Combined: $G \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright N \triangleright N_1 \triangleright N_2 \cdots \triangleright \{1\}.$

Ex. Any upper Δ gp, e.g. entries in $\mathbb{F}_p$.

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \text{is solvable}$$

$$\triangledown \qquad \mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times$$

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

$$\triangledown \qquad \mathbb{F}_p \times \mathbb{F}_p$$

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$$\triangledown \qquad \mathbb{F}_p$$

$$\left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$$

Conseq:

$$\underbrace{G \triangleright G_1}_{} \triangleright G_2 \triangleright \cdots \triangleright \{1\}$$

abelian subquotients. $= G_i/G_{i+1} \Rightarrow G$ solvable.

Non solvable gps.: $A_5 \leftarrow$ simple.

Any simple gp other than $\mathbb{Z}/p_i\mathbb{Z}$

S_5 . & S_n $n \geq 5$.

Next week: A number is a nested radical

$\Updownarrow$

its Gal. gp (Galgp of its min poly) is Solvable.