Breaking extensions into pieces

$F \subset K$ Galois

$$\underbrace{F \subset L}_{\text{Galois}} \underbrace{\subset K}_{\text{Galois}}$$

$G = \mathrm{Aut}(K/F)$
$H = \mathrm{Aut}(K/L)$.

$$\left.\begin{array}{l} ① \ F \subset L \text{ Galois} \\ ② \ \sigma(L) = L \quad \text{for all } \sigma \in G \\ ③ \ H \subseteq G \text{ normal.} \end{array}\right\} \text{Eqv.}$$

① ⇒ ②: Take $\alpha \in L$ want: $\sigma(\alpha) \in L$

Let $p(x) \in F[x]$ be the min poly of α. Then $p(x)$ splits completely over L. (because $F \subset L$ is Galois). & $\sigma(\alpha)$ also satisfies $p(x) = 0$. $\Rightarrow \sigma(\alpha) \in L$.

② ⇒ ③ Take $h \in H$ and $\sigma \in G$. Want: $\sigma h \sigma^{-1} \in H$
Take $\alpha \in L$ Then $\sigma h \underbrace{\sigma^{-1}(\alpha)}_{\text{in } L} = \sigma \sigma^{-1}(\alpha) = \alpha$.

③ ⇒ ① Take $\alpha \in L$. ~~Othe~~ All roots of min poly of α are $\{\sigma\alpha \mid \sigma \in G\}$

Want: $\sigma\alpha \in L$. Take $h \in H$ want: $h(\sigma\alpha) = \sigma\alpha$

$$\underbrace{\sigma^{-1} h \sigma}_{\text{in } H}(\alpha) = \alpha$$

□

(2) $\sigma(L) = L$

$\Rightarrow$ Can Take $\sigma : K \to K$ & restrict its to L

$\sigma : L \to L$

Gives $\mathrm{Aut}(K/F) \longrightarrow \mathrm{Aut}(L/F)$

with kernel $= \mathrm{Aut}(K/L)$

$\Rightarrow$ Image has size $|\mathrm{Aut}(K/F)| / |\mathrm{Aut}(K/L)| = |\mathrm{Aut}(L/F)|$

$\Rightarrow$ Image is $\mathrm{Aut}(L/F) \Rightarrow$ surj.

First iso thm $\Rightarrow$ $\mathrm{Aut}(K/F) / \mathrm{Aut}(K/L) \cong \mathrm{Aut}(L/F)$.

<u>Quartic</u>: Take $p(x) \in F[x]$ K = Splitting field.

$$\underset{\substack{\uparrow \\ \text{coeff}}}{F} \subset \underset{\substack{\uparrow \\ \text{roots.}}}{K}$$

$G = \mathrm{Aut}(K/F) \hookrightarrow S_4$

Assume $G = S_4$ (hardest case).

$\zeta_2, \zeta_3 \in F$.

Try $A_4 \subset S_4$

$$\underbrace{F \overset{C_2}{\subset} L_1}_{2} \; \underbrace{\overset{A_4}{\subset} K}_{12}$$

$$\left.\begin{array}{l} \mathrm{id} \\ (12)(34) \\ (13)\ (24) \\ (14)\ (23). \end{array}\right. \triangleleft A_4$$

K_4

$$\underbrace{F \subset L_1}_{C_2} \underbrace{\subset L_2}_{C_3} \underbrace{\subset K}_{K_4}$$

$K_4 = C_2 \times C_2$

$C_2 \triangleleft K_4 \twoheadrightarrow C_2$

$$\underbrace{F \subset L_1}_{C_2} \underbrace{\subset L_2}_{C_3} \underbrace{\subset L_3}_{C_2} \underbrace{\subset K}_{C_2} \quad \checkmark$$

Suppose $\underbrace{F \subset K}$ is Galois with Galois gp G.

$\exists \quad F \subset L_1 \subset L_2 \subset \cdots \subset L_k = K$ $\iff$ $\underbrace{G \rhd H_1}_{C_{p_1}} \underbrace{\rhd H_2}_{C_{p_2}} \rhd \cdots\cdots \rhd 1$

$L_i \subset L_{i+1}$ is Galois of order p_i
$= \text{gp } \mathbb{Z}/p_i\mathbb{Z}$
$= C_{p_i}$

$(S_4 \rhd A_4 \rhd K_4 \rhd C_2 \rhd 1)$

A finite group G is solvable if $\exists$ chain of subgroups

$$G \rhd G_1 \rhd G_2 \rhd G_3 \rhd \cdots\cdots \rhd 1$$

$G_{i+1} \subset G_i$ is normal & $G_{i+1}/G_i \cong \mathbb{Z}/p_i\mathbb{Z}$ prime p_i

If $p(x) \in F[x]$ has a solvable Galois gp (& F contains enough roots of 1) then the roots of $p(x)$ are expressible as iterated radicals of elts in F.