Writing using radicals $\leftarrow \sqrt[p]{}$ "Surd"

Take $\mathbb{C} \supset F$ a subfield.

Say $\alpha \in \mathbb{C}$ is expressible using radicals over F if $\exists$ a chain of fields

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \ni \alpha$$

(p_i primes).

where $F_1 = F_0[\zeta_{p_1}, \ldots, \zeta_{p_k}]$ and

$F_i \subset F_{i+1}$ is (cyclic) Galois of ~~order~~ deg p for some $p \in \{p_1, \ldots, p_k\}$.

$\Leftrightarrow$ $F_{i+1} = F_i[\sqrt[p]{a_i}]$ for some $a_i \in F_i$

Ex. $\alpha = \sqrt[3]{1+\sqrt[5]{2}}$ exp. using $\sqrt{\ }$ over $\mathbb{Q}$.

$$\mathbb{Q} = F_0 \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset \underset{\displaystyle F_1[\sqrt[5]{2}]}{\overset{\|}{F_2}} \subset \underset{\displaystyle F_2[\sqrt[3]{1+\sqrt[5]{2}}]}{\overset{\|}{F_3}} \ni \alpha$$

Ex. $1 + \sqrt[5]{1+\zeta_3^2}$

$$F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset F_2 = F_1[\sqrt[5]{1+\zeta_3^2}] \ni \alpha$$

Thm. (Abel-Galois). α is exp. using radicals over F iff $\underbrace{\text{the Galois gp of } \alpha \text{ over } F}_{\text{Galois gp of } \underbrace{\text{min poly of } \alpha \text{ over } F}_{p(x)}}$ is solvable.

$F \subset K =$ Splitting field of $p(x)$.

$$\boxed{\begin{gathered}(x^5 - (1+\zeta_3^2)) \cdot \\ (x^5 - (1+\zeta_3))\end{gathered}}$$

Pf α exp using radicals $\Longrightarrow$ Solvable.

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \ni \alpha$$

$F \subset F_n$ need to be Galois!

Ex. $\mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset F_2 = F_1[\sqrt[5]{1+\zeta_3^2}]$

$$F_2' = F_1[\sqrt[5]{1+\zeta_3^2}, \sqrt[5]{1+\zeta_3}] \ni \alpha$$

Galois

Solvable Galois gp.

$$\mathbb{Q} \subset \mathbb{Q}[\zeta_3, \zeta_5] \subset \mathbb{Q}[\zeta_3, \zeta_5][\sqrt[5]{1+\zeta_3^2}, \sqrt[5]{1+\zeta_3}]$$

Prop: F field char 0 $f(x), g(x)$ two poly.

$F \subset K$ = Splitting field of $f(x) \cdot g(x)$

$K \supset K_1$ = Splitting field of $f(x)$

$ \supset K_2$ = Splitting field of $g(x)$

$$\begin{tikzcd} & K & \\ K_1 \arrow[ur, phantom, "\subset"] & & K_2 \arrow[ul, phantom, "\supset"] \\ & F \arrow[ul, phantom, "\supset"] \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

$$\mathrm{Aut}(K/F) \longrightarrow \mathrm{Aut}(K_1/F) \times \mathrm{Aut}(K_2/F) \quad \text{hom.}$$

is injective.

Pf: Say $\sigma \in \mathrm{Ker}$. $\Rightarrow$ σ fixes all roots of $f(x)$
fixes all roots of $g(x)$

$\Rightarrow$ σ fixes all in K.

Cor: Gal. gp of $f(x) \cdot g(x) \cong$ Subgp of $\mathrm{Gal}(f(x)) \times \mathrm{Gal}(g(x))$.

$$\underbrace{\mathbb{Q} \subset \mathbb{Q}[\zeta_3, \zeta_5]}_{\text{sub. of } C_2 \times C_4} \subset \underbrace{\mathbb{Q}[\zeta_3, \zeta_5][\sqrt[5]{\ldots}, \sqrt[5]{\ldots}]}_{\text{sub. of } C_5 \times C_5} \ni \alpha$$

← both solvable

Solvable.

$\mathbb{Q} \subset F_2$ ← Galois

$\alpha \in F_2$

$\Rightarrow$ all roots of min poly (α) $\in F_2$

we're after.

$$\underbrace{\overbrace{\mathbb{Q} \subset K}^{\text{we're after}} \subset F_2}_{\text{solvable}}$$

$K = \mathbb{Q}[\quad\quad]$

solvable

$\Rightarrow$ $Gal(K/\mathbb{Q})$ is ~~the~~ quotient of a solv. gp.

$\Rightarrow$ solvable.