Solvability.

α exp. as radicals $\Rightarrow$ $Gal(\alpha)$ is solvable.

$\downarrow$

~~$F_0 = \mathbb{Q}$~~ $F = \underbrace{F_0 \subset F_1}_{\text{roots of } 1} \subset F_2 \subset \cdots \subset F_n \ni \alpha$ $\qquad F_{i+1} = F_i[\sqrt[p_i]{a_i}]$

Goal: Construct $F_n' \supset F_n$ such that $F \subset F_n'$ is Galois with Solvable Galois group.

Strategy: Do this for $F_i' \supset F_i$ inductively, starting with $i = 1$.

$F_1' = F_1$. Suppose F_i' has been constructed.

$$\begin{array}{ccc} F & \subset & F_i \\ & \searrow & \cap \\ & & F_i' \end{array}$$

Solvable Galois gp G_i

$F_{i+1} = F_i[\sqrt[p]{a_i}]$ for some prime p, $a_i \in F_i$

Let

$$F_{i+1}' = F_i'[\sqrt[p]{g a_i} \mid g \in G_i]$$

F_{i+1}' is the splitting field over F of $\prod_{g \in G_i} (x^p - g a_i) \cdot h(x)$

where F_i' is the splitting field of $h(x)$ over F.
(i.e.g. take h = product of min poly of any set of generators of F_i' over F).

$\Rightarrow$ $F \subset F_{i+1}'$ is Galois.

Why solvable?

$$\underbrace{F \subset F_i'}_{G_i} \subset \underbrace{F_i' \subset F_{i+1}'}_{\text{subgp of } \prod \mathbb{Z}/p\mathbb{Z}}$$

$\Downarrow$

$Gal(f(x) \cdot g(x)) \subset Gal(f(x)) \times Gal(g(x))$.

$\Rightarrow$ $Gal(F_{i+1}'/F_0)$ is solvable.

Gives $F_n' \supset F$ solvable Gal. gp. $\alpha \in F_n'$

$$\underbrace{F_n' \supset \text{Splitting field of min poly of } \alpha \supset F}_{\text{solvable}} \Longrightarrow Gal(\alpha) \text{ is solvable.}$$

$\square$

$$\overbrace{F \subset F[\zeta_p, \dots]}^{\text{Solvable.}} \subset K[\zeta_p \dots] \Downarrow$$

$$F[\zeta_p, \dots] = F_1$$

$$\underbrace{F[\zeta_p, \dots] \subset K[\zeta_p \dots]}_{\text{solvable.}}$$

$K[\zeta_p \dots]$: splitting field of $f(x)$ over F_1

solvable.

$\Downarrow$

composition series.

$\Downarrow$

$$F_1 \subset F_2 \subset F_3 \subset \dots \subset K[\zeta_p \dots]$$

$\hookrightarrow$ primes have to divide

$$\left| \underset{\text{over } F_1}{Gal(p(x))} \right| \;\Big|\; n!$$

$\Rightarrow \zeta_p$ already in F_1.

$\Rightarrow F_{i+1} = F_i[\sqrt[p]{a_i}]$.

by Kummer.

$\square$

Cor: No ~~Any~~ root of X^5+2x+2 is expressible by radicals over $\mathbb{Q}$.

Converse. If $\mathrm{Gal}(\alpha)$ is solvable $\Rightarrow$ α is exp. by radicals.

Let $f(x)$ be min poly of α over F (think $F=\mathbb{Q}$).

Take $F_1 = F[\zeta_p$ for all $p \leq n!]$ $n=\deg f(x)$.

What's Gal. gp. of $f(x)$ over F_1? (Also solvable). reason.

$\Leftarrow$ $K=$ splitting field of $f(x)$ over F

$K[\zeta_p|\ldots] =$ Splitting field of $f(x)\cdot\prod(x^p-1)$ over F

$$\underbrace{F \subset K}_{\text{solvable}} \underbrace{\subset K[\zeta_p\ldots]}_{\text{abelian.}} \Rightarrow \underbrace{F \subset K[\zeta_p\ldots]}_{\text{solvable.}}$$