FINAL EXAM

DIFFERENTIAL FORMS IN ALGEBRAIC TOPOLOGY, 2024

1. Instructions

(1) You may consult *Differential forms in algebraic topology* by Bott and Tu. You may not consult any other source, including any source on the internet. You may not contact any other students in the course during the exam.

(2) Some students are taking the exam at a slightly different time, so please do not talk about the exam with anyone until the end of the week.

(3) You may use any result from the book or the homework. If you use a result from the book, please cite it by name or number. If you use a result from the homework, please clearly write the statement that you are using.

For example, the following are OK:

(a) "By the Thom isomorphism theorem ..."

(b) "By Theorem 6.8 from Bott and Tu ..."

(c) "On the homework, we proved that $\mathbf{RP}^n$ is orientable if n is odd. Therefore, ..."

For example, the following are not OK:

(a) "By a result from Bott and Tu ..."

(b) "By a homework problem, ..."

(4) There are four questions, each worth 10 marks.

(5) Write clearly and legibly on any paper (typing is OK too, but not required). Please scan and send me your exam by email by 9pm.

(6) If you have questions, feel free to ask by email.

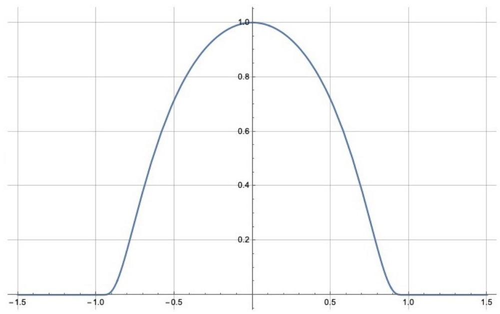
The questions are on the next page.

Date: 13 November 2024.

2. Questions

Please justify your answers unless specifically asked otherwise.

(1) Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function with the following graph.



Note that f is smooth, compactly supported, and $\int_{\mathbf{R}} f(x)dx = 1$. Let x, y be the standard coordinates on $\mathbf{R}^2$. Set $\omega = f(x)f(y)dx \wedge dy$.

(a) Does there exist a 1-form η such that $\omega = d\eta$? If so, find η. If not, why not?

(b) Does there exist a compactly supported 1-form η such that $\omega = d\eta$? If so, find η. If not, why not?

(2) Let $L \subset \mathbf{R}^3$ be a line and $p \in \mathbf{R}^3$ a point not on L. Set $M = \mathbf{R}^3 - (L \cup \{p\})$ (the complement of the union of L and p).

(a) Find $H_c^*(M, \mathbf{R})$.

(b) Describe closed submanifolds of M whose Poincare duals form a basis of $H_c^*(M, \mathbf{R})$. Justify your answer for $H_c^1(M, \mathbf{R})$ **or** $H_c^2(M, \mathbf{R})$.

(3) Let $E \to S^1$ be a non-trivial real vector bundle of rank 2. Which of the following are trivial: E^*, $E \oplus E$, $E \otimes E$.

(4) Consider the standard open cover $\{U_0, U_1, U_2\}$ of $\mathbf{CP}^2$. That is,

$$U_i = \{[x_0 : x_1 : x_2] \mid x_i \neq 0\}.$$

Consider the Cech–de-Rham double complex for this open cover with coefficients in $\mathbf{R}$.

(a) Write the E_1 page of the associated spectral sequence.

(b) Compute the E_2 page.