Midterm practice questions

Algebra 2

2026 Semester 1

1. Let R be a domain and $r\colon R \setminus \{0\} \to \mathbf{Z}_{\geq 0}$ a function. What does it mean when we say that r makes R a Euclidean domain?

   Let $R = \mathbf{Z}[\sqrt{-7}]$ and let $r(z) = |z|^2$. Does r make R a Euclidean domain? Why, or why not?

2. Define a principal ideal domain. Give an example and a non-example.

3. Define a unique factorisation domain. Give an example and a non-example. In your non-example, find an irreducible element that is not prime.

4. Find the prime factorisation of 30 in $\mathbf{Z}[i]$. For each prime p in the prime factorisation, describe the ring $\mathbf{Z}[i]/(p)$. In particular, find its characteristic and its cardinality. Is $\mathbf{Z}[i]/(30)$ an integral domain? Does $\mathbf{Z}[i]/(30)$ have nilpotent elements? (A *nilpotent* element is an non-zero element r such that $r^n = 0$ for some positive integer n.)

5. Let p be prime. Is the polynomial
$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!}$$
   irreducible in $\mathbf{Q}[x]$?

6. Find the kernel of the map $\mathbf{Q}[x] \to \mathbf{C}$ that sends x to $i + 2$.

7. Find the kernel of the map $\mathbf{Z}[x] \to \mathbf{R}$ that sends x to $1/3$.

8. Let F and K be fields with $F \subset K$.

   (a) Define the degree of K over F.

   (b) Let $\alpha \in K$. How is the degree of α over F related to
   - the degree of K over F?
   - the degree of $F(\alpha)$ over F?

9. Let $F \subset K$ be a field extension of prime degree.

   (a) Prove that for any $a \in K$ such that $a \notin F$, we have $K = F(a)$.

   (b) Give a counter-example to this statement when the degree is not prime.

10. Construct a field with 9 elements and one with 125 elements. Is there a field with 28 elements? Why?

11. Let p be a prime number.

    (a) What is the degree of $e^{2\pi i/p}$ over $\mathbf{Q}$? Over $\mathbf{R}$?

    (b) What is the degree of $e^{\pi i/p}$ over $\mathbf{Q}$?

12. Can the point $(2^{1/3}, 2^{-1/3})$ be constructed with a ruler and compass starting only from $(0, 0)$ and $(1, 0)$?

13. Let p be a prime number. Prove that a regular p-gon can be constructed with a ruler and compass, starting with the points $(0, 0)$ and $(1, 0)$, only if $p = 2^n + 1$ for some positive integer n.

14. Prove that $x^3 + t^3x^2 + t \in \mathbf{C}[x, t]$ is irreducible.

15. Let $F \subset K$ be a field extension.

(a) Define a *primitive element* of K over F.

(b) Give an example and a non-example.

(c) Find a primitive element of $\mathbf{Q}[2^{1/3}, e^{2\pi i/3}]$.

16. Can $\sqrt{2}$ be written as a polynomial in $\sqrt{2} + \sqrt{5}$ with rational coefficients?

17. Describe all maximal ideals of $\mathbf{R}[x]$.

18. Let $F = \mathbf{F}_2[a]/(a^4 + a + 1)$ (this is a field).

(a) Find the degree of a^2 over $\mathbf{F}_2$.

(b) How many elemenst of F have degree 2 over F_2?

(c) Find them.

(d) Is there an isomorphism between F and $\mathbf{F}_2[b]/(b^4 + b^2 + 1)$? If not, why not. If yes, find one/all.