

Workshop 7

Algebra 2

2026 Semester 1

Symmetric polynomials

1. Write the following polynomials in terms of elementary symmetric polynomials: $x^2 + y^2$, $x^3 + y^3$, and $x^2 + y^2 + z^2$.

The elementary symmetric polynomials in x, y are $x + y, xy$. Then

$$x^2 + y^2 = (x + y)^2 - 2xy.$$

Also,

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

The elementary symmetric polynomials in x, y, z are $x + y + z, xy + xz + yz$ and xyz . Then

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz)$$

2. Most computer algebra systems have in-build algorithms to rewrite symmetric polynomials in terms of elementary symmetric polynomials. For example, the following sage code

```
S.<x,y,z> = PolynomialRing(QQ, 3)
E = SymmetricFunctions(QQ).elementary()
E.from_polynomial((x*y-z)*(x*z-y)*(y*z-x)) #Modify for your polynomial.
```

outputs

```
e[2, 2] - e[3] - 2*e[3, 1] - e[3, 1, 1] + 2*e[3, 2] + e[3, 3] +
2*e[4] + e[4, 1] - 2*e[4, 2] - 5*e[5] + 2*e[5, 1] - 2*e[6]
```

Explanation of notation:

- $e[i]$ is the i -th elementary symmetric polynomial e_i .
- $e[i, j]$ is $e_i \cdot e_j$, and so on.
- Since we only have 3 variables, e_4 and above are zero. So, the expression `sage` gave reduces to

$$e_2^2 - e_3 - 2e_3e_1 + e_3e_1^2 + 2e_3e_2 + e_3^2.$$

3. Try the code above with other symmetric polynomials. You can evaluate `sage` code online at <https://sagecell.sagemath.org/>.

Splitting fields

1. Let $F \subset K$ be a field extension and $f(x) \in F[x]$ a polynomial. When is K called a “splitting field” of $f(x)$?

Let f be a polynomial with coefficients in a field F , not necessarily irreducible. A splitting field of f over F is an extension field K/F such that

- f splits completely in K , say $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ with $\alpha_i \in K$, and
- K is generated by the roots: $K = F(\alpha_1, \dots, \alpha_n)$.

2. The following extensions are splitting fields for some $f(x)$. Give an example of such an $f(x)$ (there may be more than one).

- (a) $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}]$: $f(x) = x^2 - 2$
- (b) $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}, \sqrt{3}]$: $f(x) = (x^2 - 2)(x^2 - 3)$
- (c) $\mathbf{C}(t^n) \subset \mathbf{C}(t)$: $f(x) = x^n - t^n$
- (d) $F_3 \subset \mathbf{Z}[i]/3$: $f(x) = x^2 + 1$

3. Let $f(x)$ be the minimal polynomial of $\sqrt{2} + \sqrt{3}$. Is $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}, \sqrt{3}]$ a splitting field for $f(x)$?

Let $\alpha = \sqrt{2} + \sqrt{3}$.

$$\alpha^2 = 5 + 2\sqrt{6} \Rightarrow \alpha^2 - 5 = 2\sqrt{6}.$$

Squaring again:

$$(\alpha^2 - 5)^2 = 24 \Rightarrow \alpha^4 - 10\alpha^2 + 1 = 0.$$

Hence the minimal polynomial of $\sqrt{2} + \sqrt{3}$ is $f(x) = x^4 - 10x^2 + 1$.
The four roots of the minimal polynomial are

$$\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}, -\sqrt{2} - \sqrt{3}.$$

Enough to show that

$$\mathbf{Q}[\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}, -\sqrt{2} - \sqrt{3}] = \mathbf{Q}[\sqrt{2}, \sqrt{3}].$$

Evaluating expressions in terms of roots

Suppose $f(x) = x^3 + 2x - 2$ and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ be the roots of $f(x)$.

1. We can evaluate a symmetric expression in α_i by re-writing it in terms of the coefficients. For example, evaluate

$$(\alpha_1 + \alpha_2 - \alpha_3)(\alpha_1 + \alpha_3 - \alpha_2)(\alpha_2 + \alpha_3 - \alpha_1).$$

Use

$$\alpha_1 + \alpha_2 + \alpha_3 = 0,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = 2,$$

$$\alpha_1\alpha_2\alpha_3 = 2,$$

to get

$$(\alpha_1 + \alpha_2 + \alpha_3 - 2\alpha_3)(\alpha_1 + \alpha_2 + \alpha_3 - 2\alpha_2)(\alpha_1 + \alpha_2 + \alpha_3 - 2\alpha_1) = -8\alpha_1\alpha_2\alpha_3 = -16.$$

2. How do we evaluate asymmetric expressions? We cannot, exactly, but we can pin them down up to a finite ambiguity. For example, consider $\alpha = \alpha_1\alpha_2 - \alpha_3$. The S_3 orbit of the expression $x_1x_2 - x_3$ contains 2 other expressions: $x_1x_3 - x_2$ and $x_2x_3 - x_1$. Let $\beta = \alpha_1\alpha_3 - \alpha_2$ and $\gamma = \alpha_2\alpha_3 - \alpha_1$.

- (a) Prove that $(x - \alpha)(x - \beta)(x - \gamma)$ has coefficients in \mathbf{Q} .
- (b) Find these coefficients. You have then narrowed down α to 3 possible values, namely the roots of this polynomial. To do this, you will have to rewrite symmetric polynomials in terms of elementary symmetric polynomials. Do this using the `sage` code above.

(a) Let

$$\alpha = \alpha_1\alpha_2 - \alpha_3, \quad \beta = \alpha_1\alpha_3 - \alpha_2, \quad \gamma = \alpha_2\alpha_3 - \alpha_1.$$

Consider the polynomial

$$(T - \alpha)(T - \beta)(T - \gamma).$$

Its coefficients are

$$\alpha + \beta + \gamma, \quad \alpha\beta + \alpha\gamma + \beta\gamma, \quad \alpha\beta\gamma.$$

Each of these expressions is symmetric in $\alpha_1, \alpha_2, \alpha_3$, hence can be written as a polynomial in the elementary symmetric polynomials

$$e_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad e_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad e_3 = \alpha_1\alpha_2\alpha_3.$$

Since $e_1, e_2, e_3 \in \mathbf{Q}$, it follows that the coefficients of

$$(T - \alpha)(T - \beta)(T - \gamma)$$

lie in \mathbf{Q} .

(b) We compute the coefficients in terms of e_1, e_2, e_3 . First,

$$\alpha + \beta + \gamma = (\alpha_1\alpha_2 - \alpha_3) + (\alpha_1\alpha_3 - \alpha_2) + (\alpha_2\alpha_3 - \alpha_1) = e_2 - e_1.$$

Hence

$$\alpha + \beta + \gamma = 2 - 0 = 2.$$

Next,

$$\alpha\beta + \alpha\gamma + \beta\gamma = -e_1e_2 + e_1e_3 + e_2 + 3e_3.$$

Thus

$$\alpha\beta + \alpha\gamma + \beta\gamma = 0 + 0 + 2 + 6 = 8.$$

Finally,

$$\alpha\beta\gamma = -e_1^2e_3 - 2e_1e_3 + e_2^2 + 2e_2e_3 + e_3^2 - e_3.$$

Hence

$$\alpha\beta\gamma = 0 + 0 + 4 + 8 + 4 - 2 = 14.$$

Therefore,

$$(T - \alpha)(T - \beta)(T - \gamma) = T^3 - 2T^2 + 8T - 14.$$