Workshop 2

Algebra 2

2026 Semester 1

You must be able to justify (prove) every claim you make, no matter how “trivial” it seems. If you cannot justify it, please ask me or your demonstrator. If you can justify it but are not satisfied with the proof (too long, too messy, or something else), please ask. It is possible that there is a better way, and you will benefit from knowing it.

Warm-up

1. True or false?
   (a) The element x is a prime element of $\mathbf{Q}[x,y]$.
   (b) The element x is an irreducible element of $\mathbf{Q}[x,y]$.
   (c) The element $2x+1$ is a prime element of $\mathbf{Z}[x]$.
   (d) The element $2x+1$ is an irreducible element of $\mathbf{Z}[x]$.
2. Recall the definitions of ring homomorphism, kernel, image, and the first isomorphism theorem.
3. Let $R = \mathbf{Z}/p\mathbf{Z}$. Show that the polynomial $x^p - x$ evaluates to zero for all $x \in R$. Is $x^p - x = 0$ in $R[x]$?

Irreducibility

1. Consider $z - xy \in \mathbf{C}[x,y,z]$. Show that it is irreducible in two ways:
   (a) Identify the quotient $\mathbf{C}[x,y,z]/(z-xy)$ and conclude that $z - xy$ is prime.
   (b) Show that it is primitive as an element of $\mathbf{C}[x,y][z]$ and irreducible in $\mathbf{C}(x,y)[z]$.
2. Suppose $f(x) \in \mathbf{Z}[x]$ is irreducible. Does this imply that $f(x) \in \mathbf{Q}[x]$ is also irreducible? Is the converse true?
3. Suppose $f(x) \in \mathbf{Z}[x]$ is irreducible. Does this imply that the image of $f(x)$ in $\mathbf{Z}/p\mathbf{Z}[x]$ is also irreducible? Is the converse true? (Be careful!)
4. (Carried over from last week) Factor $x^4 + 2x^3 - x^2 + 2x - 1$ in $\mathbf{F}_2[x]$ and $\mathbf{F}_3[x]$. What does the factorisation imply about the factorisation in $\mathbf{Z}[x]$? In $\mathbf{Q}[x]$?
5. Fill in the gaps in the following argument.

   Theorem. The polynomial $f(x,y) = y^2 - x^3 \in \mathbf{C}[x,y]$ is irreducible.

   Proof. It suffices to show that $f(x,y) \in \mathbf{C}(x)[y]$ is irreducible (why?). We may make a change of variables and show that $g(x,y) = f(x,xy) \in \mathbf{C}(x)[y]$ is irreducible (why?). Showing $g(x,y) = x^2(y^2 - x)$ is irreducible in $\mathbf{C}(x)[y]$ is equivalent to showing $h(x,y) = y^2 - x$ is irreducible in $\mathbf{C}(x)[y]$ (why?), and this is in turn equivalent to showing that $y^2 - x$ is irreducible in $\mathbf{C}[x,y]$ (why?). But $y^2 - x$ is prime and hence irreducible in $\mathbf{C}[x,y]$ (why?).
6. Generalise the argument above to prove that if $\gcd(m,n) = 1$, then $y^m - x^n \in \mathbf{C}[x,y]$ is irreducible.