Workshop 4

Algebra 2

2026 Semester 1

Warm-up

1. Recall the notion of the degree of a field extension.
2. What are the degrees of the following extensions: $\mathbf{R} \subset \mathbf{C}$; $\mathbf{Q} \subset \mathbf{Q}[\zeta_5]$.
3. In each case, write a basis of the bigger field as a vector space over the smaller field.
4. Let F and K be fields and let $\phi\colon F \to K$ be a ring homomorphism. Prove that ϕ must be injective. (It is a requirement that in a field, we have $0 \neq 1$.)

Degree of a field extension

1. Let $\gamma = \sqrt{2} + \sqrt{3}$. Prove that $\mathbf{Q}[\gamma] = \mathbf{Q}[\sqrt{2}, \sqrt{3}]$. Call this field K.
2. Prove that $1, \gamma, \gamma^2, \gamma^3$ and $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are both bases of K as a $\mathbf{Q}$-vector space. Write down the change-of-basis matrix.
3. Suppose $\alpha, \beta \in \mathbf{C}$ have degrees m and n over $\mathbf{Q}$. Is the degree of the field extension $\mathbf{Q} \subset \mathbf{Q}[\alpha, \beta]$
   (a) ... equal to mn?
   (b) ... divisor of mn?
   (c) ... less than or equal to mn?

   Hint: Consider the example $\alpha = 2^{1/3}$ and $\beta = 2^{1/3}e^{2\pi i/3}$.
4. Let $\alpha = 2^{1/3}$ and $\beta = 2^{1/3}e^{2\pi i/3}$. Consider the map $\phi\colon \mathbf{Q}[x, y] \to \mathbf{Q}[\alpha, \beta]$ that sends x to α and y to β. Find the kernel of the map.

   Hint: First find the degree of $\mathbf{Q}[\alpha, \beta]$ over $\mathbf{Q}$.

Ruler and compass constructions

Let us prove that an angle whose cosine and sine are transcendental cannot be trisected.

1. Let θ be a real number. Prove that $\cos\theta$ is transcendental over $\mathbf{Q}$ if and only if $\sin(\theta)$ is transcendental over $\mathbf{Q}$.
2. Let θ be an angle such that $\cos\theta$ is transcendental over $\mathbf{Q}$. Suppose we are given the points $(0, 0), (0, 1)$, and $(\cos(\theta), \sin(\theta))$ in the plane. Prove that it is impossible to construct $(\cos(\theta/3), \sin(\theta/3))$ using only ruler and compass.

   Hints: Let $F \subset \mathbf{C}$ be the field $\mathbf{Q}(\cos(\theta))$. Construct an isomorphism $\mathbf{Q}(t) \to F$ (the first one is the field of rational functions in one variable t). Prove that the field extension $F \subset \mathbf{Q}(\cos(\theta/3))$ has degree 3 by explicitly constructing the minimal polynomial for $\cos(\theta/3)$ over F. Carefully justify why the polynomial you found is irreducible.