Workshop 4

Algebra 2

2026 Semester 1

Warm-up

1. Recall the notion of the degree of a field extension.

   Let $F \subseteq K$ be fields. The degree of K over F is defined as the dimension of K as a vector space over F.

2. What are the degrees of the following extensions: $\mathbf{R} \subset \mathbf{C}$; $\mathbf{Q} \subset \mathbf{Q}[\zeta_5]$.

   $[\mathbf{C} : \mathbf{R}] = 2$, $[\mathbf{Q}[\zeta_5] : \mathbf{Q}] = 4$

3. In each case, write a basis of the bigger field as a vector space over the smaller field.

   Basis of $\mathbf{C}$ over $\mathbf{R}$: $\{1, i\}$, Basis of $\mathbf{Q}[\zeta_5]$ over $\mathbf{Q}$: $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$

4. Let F and K be fields and let $\phi\colon F \to K$ be a ring homomorphism. Prove that ϕ must be injective. (It is a requirement that in a field, we have $0 \neq 1$.)

   The only ideals of the field F are $\{0\}$ and F. Therefore, $Ker(\phi) = \{0\}$ or F. But since $1 \mapsto 1$ and $0 \neq 1$, we have $1 \notin Ker(F)$ and thus $Ker(F) = \{0\}$.

Degree of a field extension

1. Let $\gamma = \sqrt{2} + \sqrt{3}$. Prove that $\mathbf{Q}[\gamma] = \mathbf{Q}[\sqrt{2}, \sqrt{3}]$. Call this field K.

   We have $\gamma^2 = 5 + 2\sqrt{6}$, thus $\sqrt{6} \in \mathbf{Q}[\gamma]$. Solve the system of equations $\gamma = \sqrt{2} + \sqrt{3}$, $\sqrt{6}\gamma = 2\sqrt{3} + 3\sqrt{2}$ to express $\sqrt{2}$ and $\sqrt{3}$ in terms of γ.

2. Prove that $1, \gamma, \gamma^2, \gamma^3$ and $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are both bases of K as a $\mathbf{Q}$-vector space. Write down the change-of-basis matrix.

   Show that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ are linearly independent. Suppose $a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}=0$. Then $(a+b\sqrt{2})+(c+d\sqrt{2})\sqrt{3}=0$. Since $1, \sqrt{3}$ are independent over $\mathbf{Q}[\sqrt{2}]$ we have, $a+b\sqrt{2}=0$, $c+d\sqrt{2}=0$. Similarly, since $1, \sqrt{2}$ are independent over $\mathbf{Q}$ we have $a=b=c=d=0$.

   Note that $\gamma=\sqrt{2}+\sqrt{3}$, $\gamma^2=5+2\sqrt{6}$, $\gamma^3=11\sqrt{2}+9\sqrt{3}$. Thus the change of basis matrix is given by

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 11 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

3. Suppose $\alpha, \beta \in \mathbf{C}$ have degrees m and n over $\mathbf{Q}$. Is the degree of the field extension $\mathbf{Q} \subset \mathbf{Q}[\alpha, \beta]$

   (a) … equal to mn?

   (b) … divisor of mn?

   (c) … less than or equal to mn?

   Hint: Consider the example $\alpha = 2^{1/3}$ and $\beta = 2^{1/3}e^{2\pi i/3}$.

   Note that $[\mathbf{Q}[\alpha,\beta]:\mathbf{Q}] = [\mathbf{Q}[\alpha,\beta]:\mathbf{Q}[\alpha]][\mathbf{Q}[\alpha]:\mathbf{Q}]$. Thus we have $[\mathbf{Q}[\alpha,\beta]:\mathbf{Q}] \leq mn$

4. Let $\alpha = 2^{1/3}$ and $\beta = 2^{1/3}e^{2\pi i/3}$. Consider the map $\phi\colon \mathbf{Q}[x,y] \to \mathbf{Q}[\alpha,\beta]$ that sends x to α and y to β. Find the kernel of the map. *Hint*: First find the degree of $\mathbf{Q}[\alpha,\beta]$ over $\mathbf{Q}$.

   Show that $\mathbf{Q}[\alpha,\beta] = \mathbf{Q}[\alpha,\zeta_3]$ where $\zeta_3 = e^{\frac{2\pi i}{3}}$, and that its degree over $\mathbf{Q}$ is 6.

   Show that x^3-2 and x^2+y^2+xy belong to the kernel, and show that the degree of $\mathbf{Q}[x,y]/(x^3-2, x^2+y^2+xy)$ over $\mathbf{Q}$ is 6.

Ruler and compass constructions

Let us prove that an angle whose cosine and sine are transcendental cannot be trisected.

1. Let θ be a real number. Prove that $\cos(\theta)$ is transcendental over $\mathbf{Q}$ if and only if $\sin(\theta)$ is transcendental over $\mathbf{Q}$.

   If α is algebraic, then α^2 and $\sqrt{\alpha}$ are algebraic. Show that $\cos(\theta)$ is algebraic over $\mathbf{Q}$ if and only if $\sin(\theta)$ is algebraic over $\mathbf{Q}$ using the identity $\sin^2\theta + \cos^2\theta = 1$.

2. Let θ be an angle such that $\cos\theta$ is transcendental over $\mathbf{Q}$. Suppose we are given the points $(0, 0), (0, 1)$, and $(\cos(\theta), \sin(\theta))$ in the plane. Prove that it is impossible to construct $(\cos(\theta/3), \sin(\theta/3))$ using only ruler and compass.

   Hints: Let $F \subset \mathbf{C}$ be the field $\mathbf{Q}(\cos(\theta))$. Construct an isomorphism $\mathbf{Q}(t) \to F$ (the first one is the field of rational functions in one variable t). Prove that the field extension $F \subset \mathbf{Q}(\cos(\theta/3))$ has degree 3 by explicitly constructing the minimal polynomial for $\cos(\theta/3)$ over F. Carefully justify why the polynomial you found is irreducible.

The minimal polynomial of $\cos(\theta/3)$ over $\mathbf{Q}(\cos\theta)$ is $4x^3 - 3x - \cos(\theta)$.