Workshop 6

Algebra 2

2026 Semester 1

Review

Here is a brief list of the main concepts we have seen so far. If you have any questions about any of them, please ask each other or your demonstrator.

Factorisation

1. Euclidean domains, principal ideal domains, unique factorisation domains.
2. Prime and irreducible elements
3. Factorisation in $\mathbf{Z}[x]$ versus $\mathbf{Q}[x]$, or more generally $R[x]$ and $(\operatorname{frac} R)[x]$.
4. Examples of $\mathbf{Z}[i]$ and $\mathbf{Z}[\omega]$.

Fields

1. Algebraic and transcendental elements; the irreducible/minimal polynomial.
2. Degree of a field extension; multiplicativity; application to constructions.
3. Constructing extensions by formally adjoining roots.
4. Finite fields: existence, uniqueness, containments, isomorphisms, Frobenius.
5. The primitive element theorem.

Field automorphisms

Let $\alpha_1, \ldots, \alpha_4 \in \mathbf{C}$ be the roots of $x^4 - 2 = 0$. Let $K = \mathbf{Q}[\alpha_1, \ldots, \alpha_4]$.

1. Prove that any $\mathbf{Q}$-automorphism of K must permute $\{\alpha_1, \ldots, \alpha_4\}$.
2. Conclude that we have an injective map $\operatorname{Aut}_{\mathbf{Q}}(K) \to S_4$.
3. What is the image of this map? That is, which permutations of the roots arise from automorphisms of the field? We will defer this question, but let us first observe that the answer is closely tied to *polynomial relations among the roots*. Suppose
$$\alpha_1 = 2^{1/4}, \quad \alpha_2 = 2^{1/4}i, \quad \alpha_3 = -2^{1/4}, \quad \alpha_4 = -2^{1/4}i.$$
Observe that we have the relation $\alpha_1 + \alpha_3 = 0$ but not the relation $\alpha_1 + \alpha_4 = 0$. Conclude that the permutation (34) cannot be in the image.
4. Observe that $K = \mathbf{Q}[2^{1/4}, i]$ and we have a $\mathbf{Q}$-isomorphism $\mathbf{Q}[x, y]/(x^4 - 2, y^2 + 1) \to K$ that sends x to $2^{1/4}$ and y to i. Use this presentation to enumerate all $\mathbf{Q}$-isomorphisms $K \to K$.
5. Now that you know the elements of $\operatorname{Aut}_{\mathbf{Q}}(K)$, describe the image of $\operatorname{Aut}_{\mathbf{Q}}(K) \to S_4$.