

Field automorphisms

Let $\alpha_1, \dots, \alpha_4 \in \mathbf{C}$ be the roots of $x^4 - 2 = 0$. Let $K = \mathbf{Q}[\alpha_1, \dots, \alpha_4]$.

1. Prove that any \mathbf{Q} -automorphism of K must permute $\{\alpha_1, \dots, \alpha_4\}$.

Let $f(x) = x^4 - 2$, and let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be its set of roots in \mathbf{C} . Let $\sigma \in \text{Aut}_{\mathbf{Q}}(K)$. Since each α_i is a root of $f(x)$, we have

$$\alpha_i^4 - 2 = 0.$$

Applying σ to both sides gives $\sigma(\alpha_i^4 - 2) = \sigma(0)$. Because σ is a field homomorphism and fixes every element of \mathbf{Q} , we have

$$\sigma(\alpha_i)^4 - 2 = 0.$$

Hence $\sigma(\alpha_i)$ is also a root of $x^4 - 2$.

2. Conclude that we have an injective map $\text{Aut}_{\mathbf{Q}}(K) \rightarrow S_4$.

From the previous part, σ determines a permutation $\pi_\sigma \in S_4$ by

$$\sigma(\alpha_i) = \alpha_{\pi_\sigma(i)}$$

for each $i = 1, \dots, 4$. This gives a map

$$\text{Aut}_{\mathbf{Q}}(K) \longrightarrow S_4, \quad \sigma \longmapsto \pi_\sigma.$$

We now show that this map is injective. Suppose

$$\pi_\sigma = \pi_\tau$$

for $\sigma, \tau \in \text{Aut}_{\mathbf{Q}}(K)$. Then

$$\sigma(\alpha_i) = \tau(\alpha_i)$$

for all $i = 1, \dots, 4$. Also, both σ and τ fix every element of \mathbf{Q} . Since the values of an automorphism on $\alpha_1, \dots, \alpha_4$ determine the automorphism completely, we have $\sigma = \tau$.

3. What is the image of this map? That is, which permutations of the roots arise from automorphisms of the field?

The answer is closely tied to *polynomial relations among the roots*.
 Suppose

$$\alpha_1 = 2^{1/4}, \quad \alpha_2 = 2^{1/4}i, \quad \alpha_3 = -2^{1/4}, \quad \alpha_4 = -2^{1/4}i.$$

Observe that we have the relation $\alpha_1 + \alpha_3 = 0$ but not the relation $\alpha_1 + \alpha_4 = 0$. Conclude that the polynomial (34) cannot be in the image.

Suppose, for contradiction, that the transposition (34) lies in the image of $\text{Aut}_{\mathbb{Q}}(K) \rightarrow S_4$. Then there exists some $\sigma \in \text{Aut}_{\mathbb{Q}}(K)$ such that

$$\sigma(\alpha_1) = \alpha_1, \quad \sigma(\alpha_2) = \alpha_2, \quad \sigma(\alpha_3) = \alpha_4, \quad \sigma(\alpha_4) = \alpha_3.$$

Now we use the polynomial relation $\alpha_1 + \alpha_3 = 0$. Applying σ , we get

$$\sigma(\alpha_1) + \sigma(\alpha_3) = 0.$$

Using the assumed action of σ , this gives

$$\alpha_1 + \alpha_4 = 0.$$

But

$$\alpha_1 + \alpha_4 = 2^{1/4} - 2^{1/4}i = 2^{1/4}(1 - i) \neq 0.$$

This is a contradiction.

4. Observe that $K = \mathbf{Q}[2^{1/4}, i]$ and we have a \mathbf{Q} -isomorphism $\mathbf{Q}[x, y]/(x^4 - 2, y^2 + 1) \rightarrow K$ that sends x to $2^{1/4}$ and y to i . Use this presentation to enumerate all \mathbf{Q} -isomorphisms $K \rightarrow K$.

Let $a = 2^{1/4}$. Using the given presentation a \mathbf{Q} -isomorphism $K \rightarrow K$ is determined by the images of a and i .

The image of a must be a root of $x^4 - 2$, so

$$a \mapsto a, \quad ia, \quad -a, \quad -ia.$$

The image of i must be a root of $y^2 + 1$, so

$$i \mapsto i \quad \text{or} \quad i \mapsto -i.$$

Thus we get 8 \mathbf{Q} -automorphisms of K .

5. Now that you know the elements of $\text{Aut}_{\mathbf{Q}}(K)$, describe the image of $\text{Aut}_{\mathbf{Q}}(K) \rightarrow S_4$.

Recall that

$$\alpha_1 = a, \quad \alpha_2 = ia, \quad \alpha_3 = -a, \quad \alpha_4 = -ia,$$

where $a = 2^{1/4}$.

We now compute the induced permutations of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$:

$\sigma(a)$	$\sigma(i)$	induced permutation
a	i	()
ia	i	(1234)
$-a$	i	(13)(24)
$-ia$	i	(1432)
a	$-i$	(24)
ia	$-i$	(12)(34)
$-a$	$-i$	(14)(23)
$-ia$	$-i$	(13)

Therefore the image of

$$\text{Aut}_{\mathbf{Q}}(K) \rightarrow S_4$$

is

$$\{(), (1234), (13)(24), (1432), (24), (12)(34), (14)(23), (13)\}.$$

This is a subgroup of S_4 of order 8, isomorphic to the dihedral group D_4 .