Workshop 7

Algebra 2

2026 Semester 1

Symmetric polynomials

1. Write the following polynomials in terms of elementary symmetric polynomials: x^2+y^2, x^3+y^3, and $x^2+y^2+z^2$.

2. Most computer algebra systems have in-build algorithms to rewrite symmetric polynomials in terms of elementary symmetric polynomials. For example, the following `sage` code

   ```
   S.<x,y,z> = PolynomialRing(QQ, 3)
   E = SymmetricFunctions(QQ).elementary()
   E.from_polynomial((x*y-z)*(x*z-y)*(y*z-x))  #Modify for your polynomial.
   ```

   outputs

   ```
   e[2, 2] - e[3] - 2*e[3, 1] - e[3, 1, 1] + 2*e[3, 2] + e[3, 3] +
   2*e[4] + e[4, 1] - 2*e[4, 2] - 5*e[5] + 2*e[5, 1] - 2*e[6]
   ```

   Explanation of notation:

   - $e[i]$ is the i-th elementary symmetric polynomial e_i.
   - $e[i,j]$ is $e_i \cdot e_j$, and so on.
   - Since we only have 3 variables, e_4 and above are zero. So, the expression `sage` gave reduces to

   $$e_2^2 - e_3 - 2e_3e_1 + e_3e_1^2 + 2e_3e_2 + e_3^2.$$

3. Try the code above with other symmetric polynomials. You can evaluate `sage` code online at `https://sagecell.sagemath.org/`.

Splitting fields

1. Let $F \subset K$ be a field extension and $f(x) \in F[x]$ a polynomial. When is K called a "splitting field" of $f(x)$?

2. The following extensions are splitting fields for some $f(x)$. Give an example of such an $f(x)$ (there may be more than one).

   (a) $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}]$
   (b) $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}, \sqrt{3}]$
   (c) $\mathbf{C}(t^n) \subset \mathbf{C}(t)$
   (d) $F_3 \subset \mathbf{Z}[i]/3$

3. Let $f(x)$ be the minimal polynomial of $\sqrt{2}+\sqrt{3}$. Is $\mathbf{Q} \subset \mathbf{Q}[\sqrt{2}, \sqrt{3}]$ a splitting field for $f(x)$?

Evaluating expressions in terms of roots

Suppose $f(x) = x^3 + 2x - 2$ and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ be the roots of $f(x)$.

1. We can evaluate a symmetric expression in α_i by re-writing it in terms of the coefficients. For example, evaluate

$$(\alpha_1 + \alpha_2 - \alpha_3)(\alpha_1 + \alpha_3 - \alpha_2)(\alpha_2 + \alpha_3 - \alpha_1).$$

2. How do we evaluate asymmetric expressions? We cannot, exactly, but we can pin them down up to a finite ambiguity. For example, consider $\alpha = \alpha_1\alpha_2 - \alpha_3$. The S_3 orbit of the expression $x_1x_2 - x_3$ contains 2 other expressions: $x_1x_3 - x_2$ and $x_2x_3 - x_1$. Let $\beta = \alpha_1\alpha_3 - \alpha_2$ and $\gamma = \alpha_2\alpha_3 - \alpha_1$.

   (a) Prove that $(x - \alpha)(x - \beta)(x - \gamma)$ has coefficients in $\mathbf{Q}$.

   (b) Find these coefficients. You have then narrowed down α to 3 possible values, namely the roots of this polynomial. To do this, you will have to rewrite symmetric polynomials in terms of elementary symmetric polynomials. Do this using the `sage` code above.