Workshop 8

Algebra 2

2026 Semester 1

The goal of the workshop is to see the main theorem of Galois theory in action in a moderately complicated example.

Let $K = \mathbf{Q}[3^{1/4}, i]$. This is the splitting field of $x^4 - 3$, and has the presentation

$$\mathbf{Q}[x, y]/(x^4 - 3, y^2 + 1) \to K,$$

where the map above sends x to $3^{1/4}$ and y to i. You should be able to prove these facts, but for now please proceed assuming them.

1. Use the presentation above to find $\mathrm{Aut}(K/\mathbf{Q})$.
2. Find an isomorphism from the dihedral group D_4 to $\mathrm{Aut}(K/\mathbf{Q})$.
3. Write the subgroup diagram for D_4: a vertex for each subgroup and an arrow for each inclusion.
4. For each group in the subgroup diagram, find the fixed field.

The main theorem of Galois theory says that these are all the fields containing $\mathbf{Q}$ and contained in K.