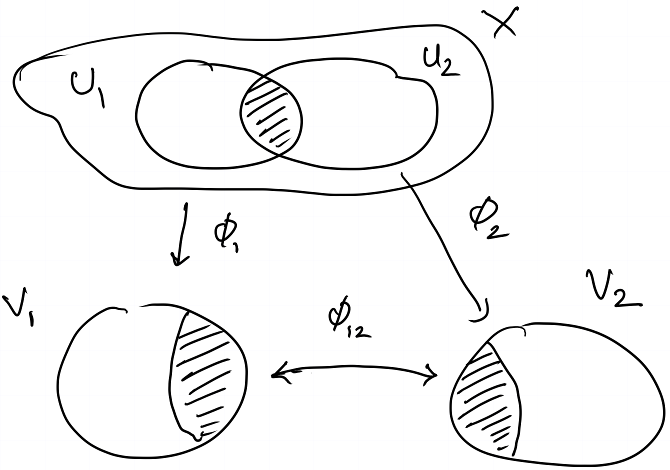
Definitions - Aug 15, 2017

Smooth manifolds.

X a topological space. A chart on X is a homeomorphism

$$\phi : U \longrightarrow V$$

where $U \subset X$ open and $V \subset \mathbb{R}^n$ open.



Two charts are C^∞-compatible if ϕ_{12} is C^∞.

A C^∞-atlas is a collection $\{\phi_i\}$ of compatible charts whose domains U_i cover X.

A smooth or C^∞ manifold is a (second countable) top. space X with a C^∞ atlas.

$$n = \dim_{\mathbb{R}} X.$$

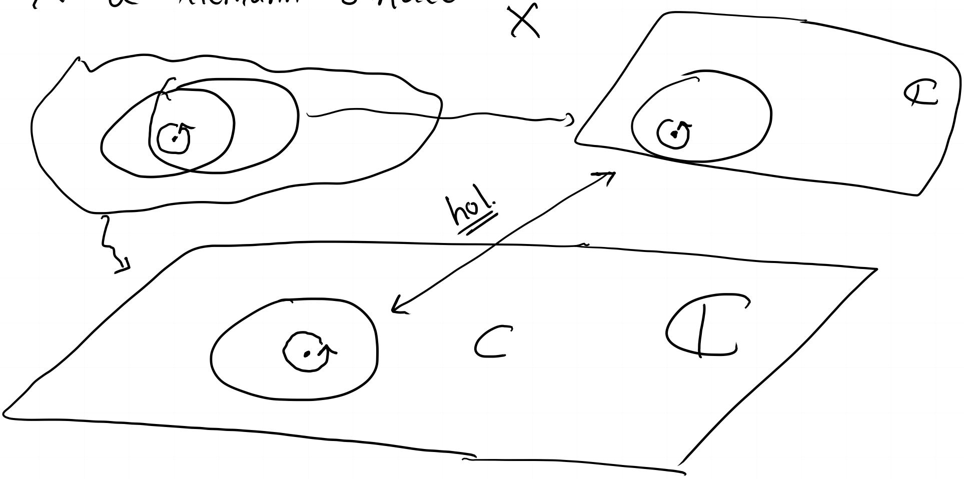
Holomorphic manifolds - Erase $\mathbb{R}$ write $\mathbb{C}$
C^∞ write holomorphic

Def: A Riemann surface is a (connected) holomorphic manifold of dim 1.

Rem: A n dimensional complex manifold is a $2n$ dim real (C^∞) manifolds.

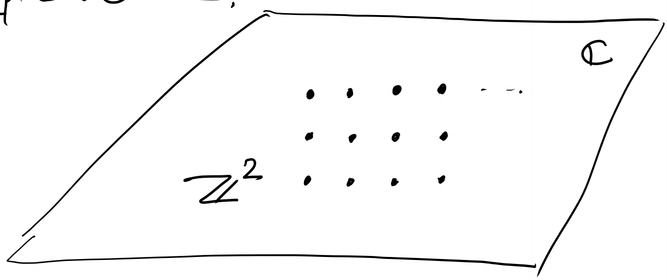
Algebraic aside: Why only $\mathbb{R}$ or $\mathbb{C}$. What about other fields?

X a Riemann surface.



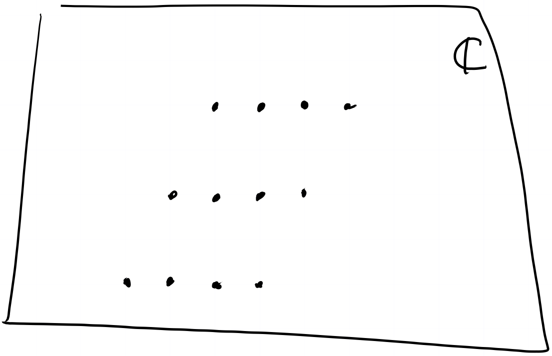
$\Rightarrow$ $\exists$ global orientation on X i.e. X is orientable.

Examples: ① $\mathbb{C}$.

② 

$\mathbb{C}/\mathbb{Z}^2 \cong S^1 \times S^1$ is a Riemann surface.

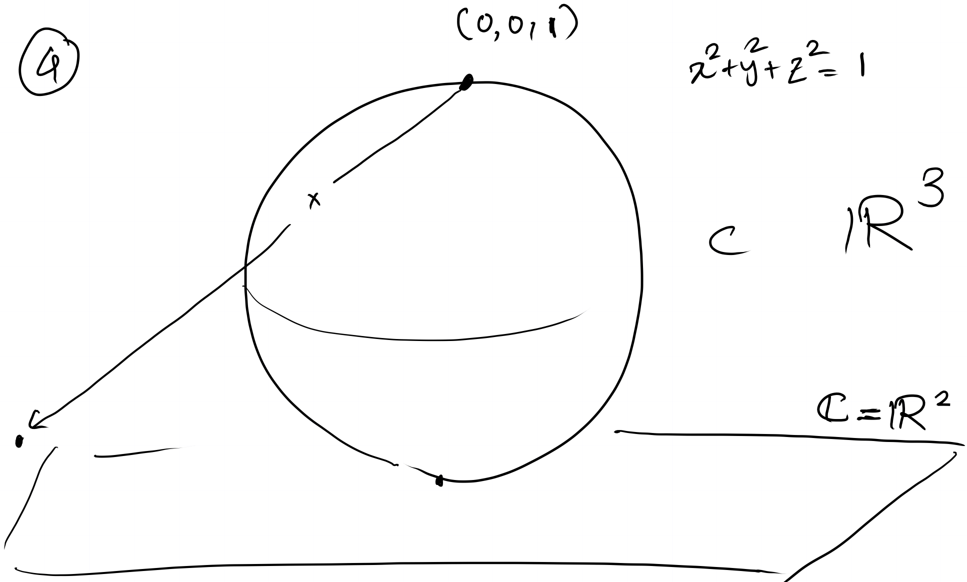
(3)



Then $\Lambda \subset \mathbb{C}$ any lattice.

$\mathbb{C}/\Lambda$ is a Riemann surface.

(4)



$U_1 = S^2 - \{(0,0,1)\}$ $\quad$ $U_2 = S^2 - \{(0,0,-1)\}$

$\phi_1 : U_1 \to \mathbb{C}$ $\Longleftarrow$ projection.

$(x,y,z) \mapsto \left(\frac{x}{1-z}\right) + \left(\frac{y}{1-z}\right) i$

$\phi_2 : U_2 \to \mathbb{C}$ $\Longleftarrow$ Projection & complex conjugation.

$(x,y,z) \mapsto \left(\frac{x}{1+z} - \frac{y}{1+z}\, i\right)$

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"] & \mathbb{C}^* \arrow[dl, dashed] \\ \mathbb{C}^* & \end{tikzcd}$$

$x = a(1-z)$

$y = b(1-z)$

$\frac{x}{1-z} = a$ $\quad x^2+y^2+z^2=1$

$\frac{y}{1-z} = b$

$a^2(1-z)^2 + b^2(1-z)^2 + z^2 = 1$

$$(x,y,z) \xleftarrow{\phi_1^{-1}} a+ib$$

$(1-z)^2 \times (a^2+b^2) = (1-z)(1+z)$

$(1-z)(a^2+b^2) = (1+z)$

$x = \frac{2a}{a^2+b^2+1}$ $\quad y = \frac{2b}{a^2+b^2+1}$ $\quad z = \frac{a^2+b^2-1}{a^2+b^2+1}$

$$\Big\downarrow \phi_2$$

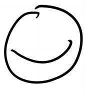
$$\left(\frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}\right)$$

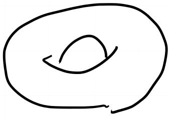
$(a+ib)$

holomophic

$\omega \mapsto \frac{1}{\omega}$

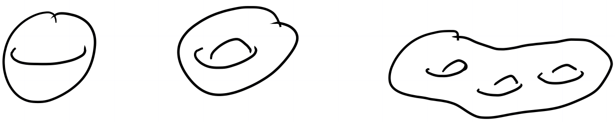
Compact Riemann Surfaces.

Ex. 1)  Riemann sphere.

2)  $\mathbb{C}/\Lambda$.

X orientable compact connected 2 manifold.

$\exists$ X is diffeomorphic to a torus with g holes



$g=0$ $\quad$ $g=1$ $\quad$ $g=3$ $\quad \cdots$

$g=$ genus of X.

Q: 1) Is there a compact RS of every genus? YES

2) How many of a given genus? Many

3) How to distinguish them?