Aug 17.

Last time - Def of Riemann surface, complex manifold
Riemann sphere - genus 0
Complex tori - genus 1

Complex projective space $\mathbb{P}^n$

$\mathbb{P}^n :=$ Set of one-dim $\mathbb{C}$ subspaces of $\mathbb{C}^{n+1}$

$= \{ (x_0, \dots, x_n) \mid x_i \in \mathbb{C} \text{ not all } 0 \} / \sim$

$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$.

$U_i \subset \mathbb{P}^n$ defined by $x_i \neq 0$

$U_i = \{ (x_0, \dots, 1, \dots, x_n) \} \cong \mathbb{C}^n$.

and $\bigcup U_i = \mathbb{P}^n$.

Topology - $U \subset \mathbb{P}^n$ open iff $U \cap U_i$ is open $\forall i$.
Charts: $\phi : U_i \to \mathbb{C}^n$.

$$\begin{array}{ccc} & U_i \cap U_j & \\ \swarrow^{\phi_i} & & \searrow^{\phi_j} \\ \mathbb{C} \times \cdots \underset{j}{\mathbb{C}^*} \cdots \times \mathbb{C} & & \mathbb{C} \times \cdots \underset{i}{\mathbb{C}^*} \cdots \times \mathbb{C} \end{array}$$

$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \dots, \frac{1}{x_j}, \dots, \frac{x_n}{x_j} \right)$
holomorphic.

Claim: $\mathbb{P}^n$ is compact
Pf: Union of $(n+1)$ boxes: $B_i \subset U_i$ def by $|x_i| \leq 1$ $\square$

Particular $\mathbb{P}^1 = \mathbb{C} \cup \mathbb{C}$ glued by

$$\phi_{12} : \mathbb{C}^* \to \mathbb{C}^*$$
$$z \mapsto \frac{1}{z}.$$

so $\mathbb{P}^1 \cong$ Riemann sphere.

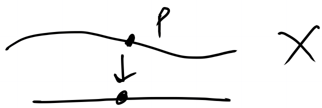
<u>Plane curves.</u>

Let $f(x,y) \in \mathbb{C}[x,y]$. Set

$$X = \{ (x,y) \in \mathbb{C}^2 \mid f(x,y) = 0 \}.$$

Suppose $\forall\, p \in X$, not both $\frac{\partial f}{\partial x}(p)$ & $\frac{\partial f}{\partial y}(p) = 0$.

Then we can make X a Riemann surface -



Suppose $\frac{\partial f}{\partial y}(p) \neq 0$ Let $x = \pi_1(p)$.

Then by the implicit function thm, $\exists$ open set $U \subset \mathbb{C}^2$ containing p and $V \subset \mathbb{C}$ containing z and a hol. function $g : V \to \mathbb{C}$ such that $X \cap U$ is the graph $\{ (z, g(z)) \mid z \in V \}$.

So $\pi_1 : X \cap U \to V$ is a chart. about p

If $\frac{\partial f}{\partial x}(p) \neq 0$, then π_2 gives a chart.

<u>Check</u>: These are compatible.

Projective curves: curves in $\mathbb{P}^2$.

$F(X,Y,Z)$ a homogeneous polynomial of deg d.

Claim: $X\dfrac{\partial F}{\partial X} + Y\dfrac{\partial F}{\partial Y} + Z\dfrac{\partial F}{\partial Z} = d\cdot F$

$$X = \{ [X:Y:Z] \mid F(X,Y,Z) = 0 \}.$$

Suppose $\dfrac{\partial F}{\partial X}$, $\dfrac{\partial F}{\partial Y}$ & $\dfrac{\partial F}{\partial Z}$ are never simultaneously 0

Then X is a Riemann surface & compact.

Why? $X \cap U_3 = \{ (x,y) \in \mathbb{C}^2 \mid \underbrace{F(x,y,1)}_{f} = 0 \}$

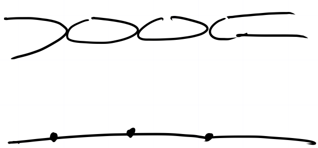
So $\forall\, p \in X \cap U_3$, $\dfrac{\partial f}{\partial x}(p)$ & $\dfrac{\partial f}{\partial y}(p)$ not both 0.

$\square$

Examples:- $f(x) \in \mathbb{C}[x]$ a polynomial.

Consider $\mathcal{U} = \{ y^2 - f(x) = 0 \} \subset \mathbb{C}^2$.
R.S. if $f(x)$ has distinct roots.

$$\begin{array}{c} \mathcal{U} \\ \downarrow \\ \mathbb{C} \end{array}$$



$\mathcal{U}$ is not compact! Can we "compactify" it?

Idea 0: Homogenize the polynomial $y^2 - f(x)$.

Ex.: $f(x) = -x^4 - 1$, $y^2 + x^4 + 1 \rightsquigarrow Z^2Y^2 + X^4 + Z^4$

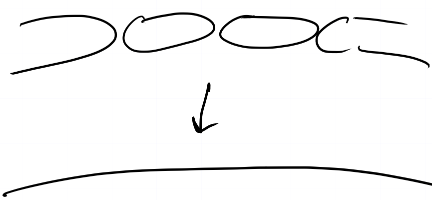
So $X = \{ Z^2 Y^2 + X^4 + Z^4 = 0 \} \subset \mathbb{P}^2$

Clearly: $X \cap U_3 = \mathcal{U}$

$$\left.\begin{aligned} \frac{\partial}{\partial Z} &= 2ZY + 4Z^3 \\ \frac{\partial}{\partial X} &= 4X^3 \\ \frac{\partial}{\partial Y} &= 2Z^2 Y \end{aligned}\right\} \quad 0 \text{ at } [0:1:0]$$

$\Rightarrow$ X is "singular" at $[0:1:0]$!

Option 1 – "Resolve" the singularities
Option 2 – do something else.

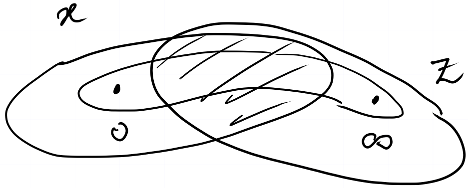


$y^2 - f(x)$

x

Try to "extend" $\mathcal{U}$ to a double cover of $\mathbb{P}^1$

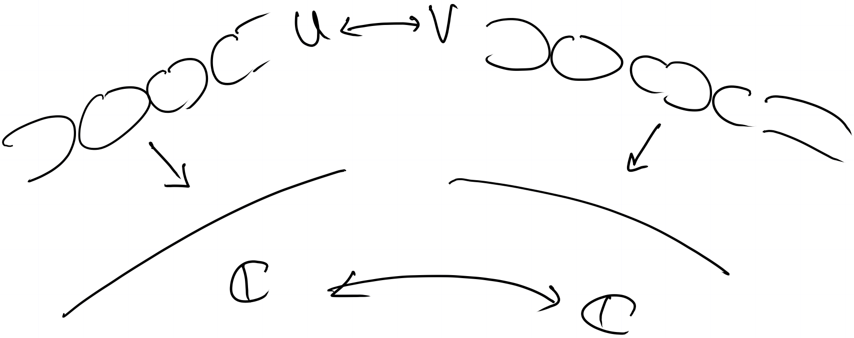
$\mathbb{C}$ $\qquad$ $\mathbb{C}$ $\qquad$ $z = \frac{1}{x}$



$$x^4 + 1 = \frac{1}{z^4} + 1 = \frac{1 + z^4}{z^4}$$

consider $g(z) = 1 + z^4$

Let $V = \{ w^2 - g(z) \} \subset \mathbb{C}^2$.



Claim $\exists$

$$\begin{array}{ccc} U^* & \xrightarrow{\sim} & V^* \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \xrightarrow{\sim} & \mathbb{C}^* \\ x & \longmapsto & \frac{1}{z} \end{array}$$

$$\{(x,y) \mid y^2 = x^4+1,\ x \neq 0\} = U^*$$

$$\{(z,w) \mid w^2 = 1+z^4,\ z \neq 0\} = V^*$$

$$(x,y) \longmapsto \left(\frac{1}{x}, \frac{y}{x^2}\right) \quad \text{so} \quad z = \frac{1}{x}, \quad w = \frac{y}{x^2}.$$

$\square$

What is the genus of X ?

$$\begin{array}{ccc} X & \supset & X^0 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \supset & \mathbb{P}^1 \setminus 4 \text{ pts} \end{array}$$

covering space.

$$\chi(X^0) = 2\,\chi(\mathbb{P}^1 \setminus 4\text{pts})$$
$$= 2\,(\chi(\mathbb{P}^1) - 4) = -4$$

So $\chi(X) = \chi(X^0) + 4$
$= 0$

so $g(X) = 0$.

<u>Generalization</u>:

Let $f(x) \in \mathbb{C}[x]$ be a poly of even degree $2n$ without multiple roots.

$$\begin{array}{c} U = \{ y^2 = f(x) \} \subset \mathbb{C}^2 \\ \downarrow \\ \mathbb{C}_x \end{array}$$

$$g(z) = f\left(\frac{1}{z}\right) \cdot z^{2n} \in \mathbb{C}[z]$$

$$\begin{array}{c} V = \{ w^2 = g(z) \} \\ \downarrow \\ \mathbb{C}_z \end{array}$$

$$\begin{array}{ccc} U^* & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ \mathbb{C}_x^* & \longrightarrow & \mathbb{C}_z^* \\ x & \longmapsto & \frac{1}{z} \end{array}$$

$(x,y) \mapsto \left(\frac{1}{x}, \frac{y}{x^n}\right)$
(z, w)

<u>Exercises</u>:
- Find $g(X)$.
- Think about what happens if $\deg f(x)$ is odd.
- Generalize to coverings $y^n = f(x)$.

Look at: $(x,y) \longrightarrow (z,w)$

$$z = \frac{1}{x}, \quad w = \frac{y}{x^n}$$

$$\begin{tikzcd}
\mathbb{C}^2_{xy} \arrow[d, "\pi_1"'] & \supset \mathcal{O} \arrow[r, leftrightarrow] \arrow[d, dash] & \mathcal{O} \subset \arrow[d] & \mathbb{C}^2_{zw} \arrow[d, "\pi_2"] \\
\mathbb{C}_x \supset & \mathbb{C}^*_x \arrow[r, leftrightarrow] & \mathbb{C}^*_z & \subset \mathbb{C}_z
\end{tikzcd}$$

Gives a surface $S \xrightarrow{\pi} \mathbb{P}^1$ "Line bundle"

Fibers of π are $\mathbb{C}$.

Def: Let X be a complex manifold. A vector bundle of rank n on X is a manifold V with a map $\pi : V \to X$ such that

① $\exists$ cover $\{U_i\}$ of X & iso ϕ_i

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[d] & U_i \times \mathbb{C}^n \arrow[dl] \\
U_i &
\end{tikzcd}$$

②
$$\begin{tikzcd}
\pi^{-1}(U_{ij}) \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & U_{ij} \times \mathbb{C}^n \arrow[d, "\phi_{12}"] \\
& U_{ij} \times \mathbb{C}^n
\end{tikzcd}$$

ϕ_{12} is linear on the fibers.

Obs: If V_1 & V_2 are trivialized on $\{U_i\}$ & have same trans functions then $V_1 \xrightarrow{\sim} V_2$.

Constructing a v.b.. Given a cover $\{U_i\}$ of X

and $\phi_{ij} : U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n$ linear on fib

$$\begin{tikzcd} U_{ij} \times \mathbb{C}^n \arrow[rr, "\phi_{ij}"] \arrow[dr] & & U_{ij} \times \mathbb{C}^n \arrow[dl] \\ & U_{ij} & \end{tikzcd}$$

such that

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik}$$

$\exists$ v.b. $V \to X$ with transition functions ϕ_{ij}.

Ex: X an n-dim manifold.

$\rightsquigarrow$ $T_X \to X$ "tangent bundle"

$\{U_i\}$ atlas for X

$\phi_{ij} : U_{ij} \to U_{ij}$ induce

ϕ_{ij} $U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n$

$(x, v) \mapsto (x, D\phi_{ij} \cdot v)$

"Cotangent bundle" = "Dual" of tangent bundle

trans. func^n = transpose $\circ$ inverse of original.

Similarly $\otimes$, $\oplus$, ...

On $\mathbb{P}^1$: $\phi_{ij} : \mathbb{C}^*_x \times \mathbb{C}_y \to \mathbb{C}^*_z \times \mathbb{C}_w$

$(x, v) \mapsto (z, z^n w)$

$\mathcal{O}(n)$ $\rightsquigarrow$

Exercise: Show $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$

$\mathcal{O}(m)^\vee = \mathcal{O}(-m)$, $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$.