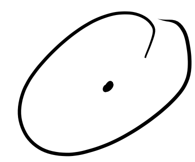
Aug 22,

X a R.S.

$\mathcal{O}_X$ = Sheaf of holomorphic functions on X.

$x \in X$ a point.

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$$



$$= \text{Germs of hol. functions at } x$$

$$\cong \text{Subring of convergent power series in } \mathbb{C}[[z]].$$

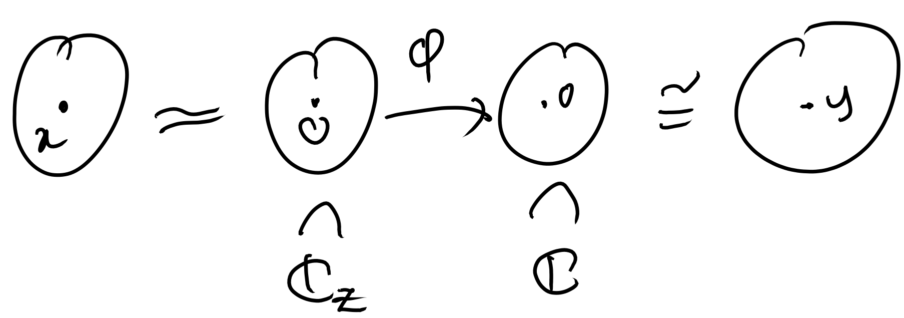
Prop: $\mathcal{O}_{X,x}$ is a local ring and a PID.
During the proof, define order of vanishing. & describe all ideals.

$\varphi : X \to Y$ a holomorphic map. Suppose $f(x) = y$.
Then we get a map

$$\varphi^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
$$f \mapsto f \circ \varphi.$$

Say $\varphi^{\#}(m_y) = m_x^n$.
The integer n is called the multiplicity or local degree of φ at x.

Concretely



$$\varphi(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad a_n \neq 0.$$

Local normal form.

$\varphi : X \to Y$, $\varphi(x) = y$ φ has mult. n at x.
Suppose $V \subset Y$ is a chart centered at y. Then $\exists$ a chart $U \subset X$ centered at x such that

$$\begin{array}{ccc} \varphi : U & \longrightarrow & V \\ \cap & & \cap \\ \mathbb{C} & & \mathbb{C} \end{array} \quad \text{is} \quad \varphi(z) = z^m.$$

Pf: Take any chart $U' \subset X$ centered at x.
Then

$\varphi : U' \to V$ will be

$$\varphi(z) = z^m T(z) \quad \text{for some hol. funct}^n\ T \text{ with } T(z) \neq 0$$

$\exists\ S(z)$ hol. near 0 such that $S(z)^n = T(z)$
Then
$$\varphi(z) = (z S(z))^n.$$

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "\psi(z) = zS(z)"'] & V \\ U \arrow[ur] & \end{tikzcd}$$

$\psi'(z) \neq 0 \Rightarrow \psi$ is a local iso (inv. f. thm).
The chart U is what we want.

$\square$.

Rem: $\varphi : X \to Y$. $\varphi(x) = y$.

$U \xrightarrow{h} V$ $\leftarrow$ Charts centered at x, y

Then $\text{mult}_x \varphi = 1 + \text{ord}_0 h'$

(Easy) Global Properties of hol. maps $F: X \to Y$

(1) F is open

(2) If F is a bijection, then F is an iso.

(3) $F, G: X \to Y$.
Then $\{ x \in X \mid F(x) = G(x) \}$ is either X or a discrete subset of X.

Thm: Let $\varphi: X \to Y$ be a hol. map between compact RS. Then the quantity

$$\sum_{x \in \varphi^{-1}(y)} \mathrm{mult}_x F$$

is indep. of y.

Def: This is called the deg of φ

Rem: The set of $x \in X$ s.t. $\mathrm{mult}_x \varphi \geq 2$ is a discrete subset of X (so finite if X compact). These are called the ramification points of φ. If y is not the image of a ram. point, then the quantity above is just the $\# \varphi^{-1}(y)$.

Pf: Let $\varphi^{-1}(y) = \{ x_1, \dots, x_m \}$ & $d_i = \mathrm{mult}_{x_i} \varphi$
By the local normal form $\exists$ charts U_i around x_i and V around y s.t.
$\varphi: U_i \to V$ is $z \mapsto z^{d_i}$. Set $U = \bigcup U_i$
We see directly that the quantity

$$\sum_{x \in \varphi^{-1}(y') \cap U} \mathrm{mult}_x \varphi$$

remains constant for y' close to y.

So it suffices to show that for y' sufficienty close to y, we have

$$\varphi^{-1}(y') = \varphi^{-1}(y') \cap U$$

i.e. $\varphi^{-1}(y') \subset U$.

Suppose the contrary. Then $\exists$ seq $y_i \to y$ and $\omega_i \in \varphi^{-1}(y_i)$ with $\omega_i \notin U$. Since X is comp., $\exists$ conv. subseq in $\{\omega_i\}$. Pass to this subseq.

Say $\lim \omega_i = \omega$. Then $\phi(\omega) = y$.

But $\varphi^{-1}(y) = \{x_1, \dots, x_m\}$

$\Rightarrow$ $\omega = x_i$ for some i.

But since U is an open set containing x_i & $\lim \omega_i = x_i$, $\omega_i \in U$ for large i.

Contradiction!

$\square$.