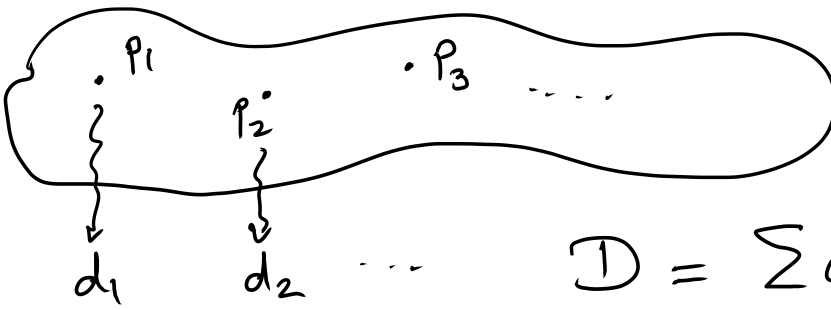
Divisors, line bundles, and all that

X a Riemann surface

A <u>divisor</u> on X is a function
$$D : X \to \mathbb{Z}$$
such that
$$\{x \in X \mid D(x) \neq 0\}$$
is discrete in X.

<u>Notation</u>: $D = \sum D(x) \cdot x$

↳ multiplicity of D at x



$$D = \sum d_i P_i$$

If X is compact, this is a finite sum.

$\mathrm{Div}(X)$ = Set of divisors on X, group under $+$

If X is compact

$\mathrm{Div}(X)$ = Free abelian group on X.

<u>Example</u> 1) f a meromorphic function on X.

$$\mathrm{Div}(f) : X \to \mathbb{Z} \qquad \sum \mathrm{ord}_x f \cdot x$$
$$x \mapsto \mathrm{ord}_x f$$

↳ "principal div"

$(fg) = (f)(g)$; $(1) = 0$; $(1/f) = -(f)$

so $f \mapsto (f)$ is a hom. $\mathcal{M}(X)^* \longrightarrow \mathrm{Div}(X)$

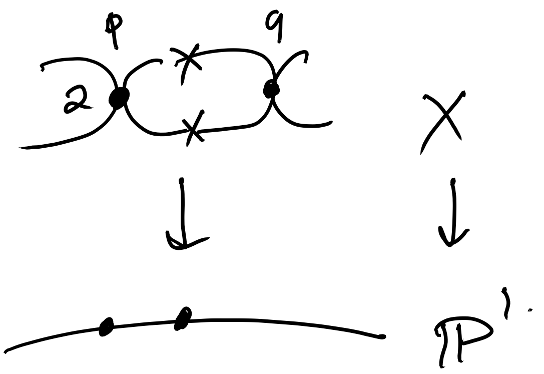
$\mathcal{M}(X)^*$ ≀ nonzero mero. fun.

Example 2: $\varphi: X \to Y$ non const.

For $y \in Y$, get $\varphi^*(y) = \sum_{x \in \varphi^{-1}(y)} (\mathrm{mult}_x \varphi) \cdot x$

More generally, for $D \in \mathrm{Div}(Y)$ we get $\varphi^*(D) \in \mathrm{Div}(X)$.

$$\varphi^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$$



Ram $= p + q + \ldots$

Example 3:

$\varphi: X \to Y$

$$\mathrm{Ram}(\varphi) = \sum_{x \in X} (\mathrm{mult}_x \varphi - 1)\, x.$$

$$\mathcal{M}(X)^* \to \mathrm{PDiv}(X) \subset \mathrm{Div}(X)$$

Def: $D_1 \sim D_2$ if $D_1 - D_2 \in \mathrm{PDiv}(X)$.
↳ Linearly equivalent. $\mathrm{Cl}(X) = \mathrm{Div}/\mathrm{PDiv}\,(X)$

Ex: $\varphi: X \to \mathbb{P}^1$.

$\varphi^*(p) \sim \varphi^*(q)$ for any $p, q \in \mathbb{P}^1$

<u>Obs</u>: $\varphi^*(f) = (f \circ \varphi)$ $\qquad X \xrightarrow{\varphi} Y$

So if $D_1 \sim D_2$ on Y

$$D_1 - D_2 = (f)$$

Then $\varphi^* D_1 - \varphi^* D_2 = (f \circ \varphi)$

so $\varphi^* D_1 \sim \varphi^* D_2$

So $\varphi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ respects lin eqv.
$\qquad\mathrm{Cl}(Y) \to \mathrm{Cl}(X)$

<u>Degree</u>: X compact.

$$\deg : \mathrm{Div}(X) \to \mathbb{Z}$$
$$\sum d_i \cdot x_i \mapsto \sum d_i$$

<u>Prop.</u>: $\deg(f) = \sum_{x \in X} \mathrm{ord}_x f = 0$

so deg descends to

$$\deg : \mathrm{Cl}(X) \to \mathbb{Z}.$$

More examples coming soon.

Line bundles.

X a complex manifold.

A <u>line bundle on X</u> is a mfld $\mathcal{L}$ & a map $\mathcal{L} \longrightarrow X$ such that

$\exists$ open cover $\{U_i\}$ of X s.t.

$$\begin{tikzcd}
U_i \times \mathbb{C} \arrow[r, "\varphi_i"] \arrow[d] & \pi^{-1}(U_i) \arrow[r] \arrow[d] & \mathcal{L} \arrow[d, "\pi"] \\
U_i \arrow[r, equal] & U_i \arrow[r] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi^{-1}(U_i \times U_j) \arrow[rr, equal] & & \pi^{-1}(U_i \cap U_j) \arrow[ddr] & & U_j \times \mathbb{C} \arrow[r, "\varphi_i"] \arrow[d] & \pi^{-1}(U_j) \arrow[dl] \\
& U_i \times U_j \times \mathbb{C} \arrow[ul, "\varphi_j"] \arrow[ur, "\varphi_i"'] \arrow[dr] & & & U_j & \\
& & U_i \cap U_j \arrow[urr, hook] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : U_{ij} \times \mathbb{C} \arrow[rr, "\varphi_{ij}", "\sim"'] \arrow[dr] & & U_{ij} \times \mathbb{C} \arrow[dl] \\
& U_{ij} \times \mathbb{C} &
\end{tikzcd}$$

φ_{ij} is fiberwise linear.

i.e. $\varphi_{ij} : (t, v) \mapsto (t, M_t v)$

$$M_t \in GL(\mathbb{C}) = \mathbb{C}^*.$$

Similarly a vector bundle of rank n.

X Riemann surface

$\mathcal{L} \xrightarrow{\pi} X$ a line bundle

Consider a "holomorphic section" $\sigma : X \to \mathcal{L}$

$$\begin{array}{ccl} \mathcal{L} & \hookleftarrow & U \times \mathbb{C} \\ \downarrow & & \downarrow \circlearrowleft \ \sigma : u \mapsto (u, s(u)) \\ X & \supset & U \qquad\qquad s : U \to \mathbb{C} \quad \text{must be hol.} \end{array}$$

If $\sigma \not\equiv 0$, we get

$$\mathrm{Div}(\sigma) = \sum_{x \in X} (\mathrm{Ord}_x \sigma) \cdot x$$

(Extends to "meromorphic sections").

Suppose σ_1, σ_2 are two meromorphic sections of $\mathcal{L}$. Then $\exists \; f \in \mathcal{M}(X)^*$ s.t.

$$\sigma_1 = f \sigma_2.$$

So $(\sigma_1) = (\sigma_2) + (f)$

| X compact $\rightsquigarrow$ degree of $\mathcal{L}$

i.e. two sections give linearly eqv. divisors.

Imp: $T_X \xrightarrow{\pi} X$ the tangent bundle

$\Omega_X \xrightarrow{\pi} X$ the cotangent bundle "canonical"

σ a mer sect. of Ω_X. (Locally $\sigma = f(z)\,dz$, f meromorphic).

$\rightsquigarrow$ (σ) called a canonical divisor.

Ex. $X = \mathbb{P}^1 = \overset{0}{\mathbb{C}_x} \cup \overset{\infty}{\mathbb{C}_y}$

$y = \frac{1}{x}$.

dx a section of Ω on $\mathbb{C}_x$

dy a section of Ω on $\mathbb{C}_y$

dx, dy. $\qquad \mathbb{C}_x^*$

so there must be a transition function

$$dy = \frac{-1}{x^2}\, dx \qquad \Big|\quad y = \frac{1}{x}, \quad dy = \frac{-1}{x^2}\, dx$$

$$dx = -x^2\, dy$$
$$= \frac{-1}{y^2}\, dy \quad \leftarrow \quad \text{mer. section of } \Omega \text{ on } \mathbb{C}_y.$$

so dx is a mer. sec of Ω on $\mathbb{P}^1$

$$(dx) = -2\cdot\infty$$

$$(dy) = -2\cdot 0$$

$$\deg(\Omega_{\mathbb{P}^1}) = -2.$$

Ex 2.

$$\begin{array}{ccc} X & \supset & U = \{z^2 - f(x) = 0\} \\ \downarrow \pi & & \downarrow \\ \mathbb{P}^1 & \supset & \mathbb{C}_x \end{array}$$

dx a hol. diff on U.

Zeros of dx. ?

x a local coord. on U if $f(x) \neq 0$. Otherwise z.

$2z\,dz = f'(x)\,dx$ so $dx = 2z\,dz / f'(x)$

$\Rightarrow$ dx has a simple zero when $f(x) = 0$

$\Rightarrow$ dx has $2n$ zeros on U.

But dx on $\mathbb{P}^1$ has a pole of order 2 at ∞
$\Rightarrow$ also a pole of order 2 at the two points over ∞

so $$(dx) = \sum_{i=1}^{2n} (0, x_i) - 2P_\infty^1 - 2P_\infty^2$$

roots of $f(x)$

so $\deg(dx) = 2n - 4 = 2g_X - 2$.

Recall $g_X = n - 1$.