Last time.

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[d] & \{\text{Line bundle} + \text{mer. sec}\} \arrow[l, "\mathrm{div}"'] \arrow[d] \\ \mathrm{Cl}(X) & \{\text{Line bundle}\} \arrow[l, "\mathrm{div}"'] \end{tikzcd}$$

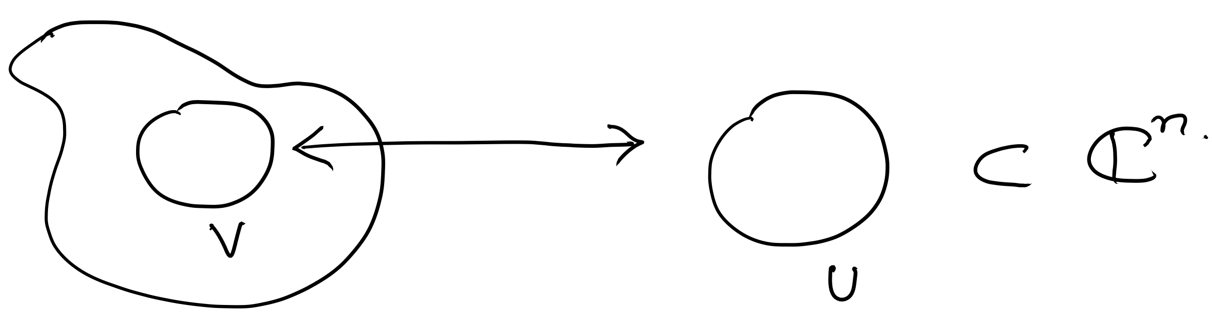
Natural bundles.

X a complex manifold.

$$T_X = \{ (x,v) \mid x \in X,\ v \in T_x X \} \xrightarrow{\pi} X$$

$U \subset \mathbb{C}^n$, $u \in U \Rightarrow T_u U = \mathbb{C}\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \rangle \cong \mathbb{C}^n$

So



$\pi^{-1}(V) \xleftrightarrow{\text{bij}} U \times \mathbb{C}^n$. Declare it to be a chart.

Thus T_X becomes a manifold and

$\pi : T_X \to X$ a vector bundle of rank n.

Similarly Ω_X

For $U \subset \mathbb{C}^n$, $u \in U \Rightarrow \Omega_{u,U} = \mathbb{C}\langle dz_1, \dots, dz_n \rangle \cong \mathbb{C}^n$.

Rem :- Operations $\oplus$, Hom, $\otimes$ on vector spaces extend to vector bundles.

$$\Omega_X = \mathrm{Hom}(T_X, \mathbb{C}_X) = T_X^*$$

f a hol. func. on X, we get a hol. section of Ω_X namely df. Locally, on a chart U on X with coord. $z_1, \dots, z_n$

$$df = \sum \frac{\partial f}{\partial z_i} \cdot dz_i$$

$$d(fg) = f\,dg + g\,df \quad , \quad d(\text{const}) = 0$$
$$d(f+g) = df + dg.$$

$$X = \mathbb{P}^1 = \mathbb{C}_x \cup \mathbb{C}_y.$$

Gives a section dx of Ω_X on $\mathbb{C}_x$.

$$\& \quad dx = \frac{-1}{y^2}\, dy.$$

So dx has a pole of order 2 at ∞.

$$\Rightarrow \quad (dx) = -2 \cdot \infty$$

so $\deg(\Omega_{\mathbb{P}^1}) = -2.$

$$\begin{array}{ccc} X & \supset & U \\ \downarrow \varphi & & \downarrow \\ \mathbb{P}^1 & \supset & \mathbb{C}_x \end{array} \qquad y^2 = f(x) \leftarrow \text{degree}$$

General rem: $\varphi: X \to Y$ hol. fun.
& ω a hol. (mer) form on Y (= section of Ω_Y).
Then we get $\varphi^*\omega$, a hol (mer) form on X (= sect. of Ω_X)

Locally:

$$\begin{array}{ccc} U & \longrightarrow & V \\ \cap & & \cap \\ \mathbb{C}_z & & \mathbb{C}_w \end{array} \qquad \varphi: z \mapsto w \qquad w = f(z)$$

and $\omega = g(w)\,dw$ then

$$\varphi^*\omega = g(f(z)) \cdot d\,f(z)$$
$$= g(f(z)) \cdot f'(z)\,dz.$$

Say $w = z^n$, $g(\omega) = c_m \omega^m + \text{h.o.t.}$

$$\varphi^*\omega = g(z^n) \cdot n\,z^{n-1}\,dz$$

$x = 0 \in U$
$y = 0 \in V.$

So $\operatorname{ord}_x(\varphi^*\omega) = n \cdot (\operatorname{ord}_y \omega) + (n-1)$

So ω has a zero of order 1 at the $2n$ ramification points of φ and poles of order 2 at the two points above ∞.

so $\deg(\omega) = 2n - 4$

Recall $g(X) = n-1$

so $\deg(\omega) = 2g - 2.$

<u>Prop</u>: Let $\varphi: X \to Y$ be a non constant map between Riemann surfaces & ω_Y a mer. diff form on Ω_Y. Let $\omega_X = \varphi^*\omega_Y$. Then

$$(\omega_X) = \varphi^*(\omega_Y) + \operatorname{Ram}\varphi$$

<u>Pf</u>: $\omega_Y = \sum \operatorname{ord}_y(\omega_Y) \cdot y$
$\varphi^*\omega_Y = \sum \operatorname{ord}_y(\omega_Y) \cdot \varphi^*(y)$

$$= \sum \text{ord}_y(\omega_Y) \cdot \sum_{x \mapsto y} (\text{mult}_x \varphi) \cdot x$$

$$= \sum_{\substack{x \in X \\ y = \varphi(x)}} (\text{mult}_x \varphi) \cdot \text{ord}_y(\omega_Y) \cdot x$$

$$\text{Ram}\, \varphi = \sum_{x \in X} (\text{mult}_x \varphi - 1) \cdot x$$

$$\text{Ord}_x \, \omega_X = \text{mult}_x \varphi \cdot \text{ord}_y(\omega_Y) + (\text{mult}_x \varphi - 1).$$

so $(\omega_X) = \varphi^*(\omega_Y) + \text{Ram}(\varphi)$.

$\square$.

<u>Cor</u>: Let X be a compact R.S. which has a non-const mer. fun. Then

$$\deg(\Omega_X) = 2g_X - 2.$$

<u>Pf</u>: $\exists \; \varphi : X \to \mathbb{P}^1$ deg d.

$(2g_X - 2) = d(2 \cdot 0 - 2) + \deg \text{Ram}$. (Rie-Hur).

$\deg \Omega_X = d \cdot (-2) + \deg \text{Ram}$. (Prop above).

so $\deg \Omega_X = 2g_X - 2$.

$\square$.

<u>Rem</u>: ① Gives a way of defining g_X.

② Turns out, all compact R.S. have non-const mer. fun!

Some more natural line bundles.

$X = \mathbb{P}^n$ = Lines in an $(n+1)$-dim v.space V.

$S \xrightarrow{\pi} X$ The "universal" line bundle.

$\|$

$\{(x,v) \mid x \in \mathbb{P}^n,\ v \in \text{Line rep. by } x\}$.

$\mathbb{P}^n \supset U = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\} \cong \mathbb{C}^n$

$$\pi^{-1}(U) \longleftrightarrow U \times \mathbb{C}$$

$$(x,v) \longleftrightarrow (x, v_i)$$

On $\mathbb{P}^1 = \{[x:1]\} \cup \{[1:y]\}$

$S = \{[x:1], [zx:z]\} \cup \{[1:y], [w:wy]\}$

$$\begin{array}{ccc} \mathbb{C}^2_{x,z} \supset \mathbb{C}^*_x \times \mathbb{C}_z & \leftrightarrow & \mathbb{C}^*_y \times \mathbb{C}_w \subset \mathbb{C}^2_{w,y} \\ & w = zx. & \\ \downarrow & & \downarrow \\ \mathbb{C}_x \supset \mathbb{C}^*_x & \leftrightarrow & \mathbb{C}^*_y \subset \mathbb{C}_y \\ & x = \frac{1}{y} & \end{array}$$

$$\begin{array}{ccc} (x,z) & \longrightarrow & [x:1],\ (zx,z) \\ \updownarrow & & \downarrow \\ (\frac{1}{x}, zx) & \longleftarrow & [1:1/x],\ (zx,z) \end{array}$$

Claim: $\deg(S) = -1$.

Pf: Consider the section: $[x:y] \mapsto [x:y], (\frac{x}{y}, 1)$.

on $\mathbb{C}_x$: $x \mapsto (x,1)$, $\mathbb{C}_y$: $y \mapsto (y, \frac{1}{y})$. $\square$

Some more generalities.

- ⊗ makes {LB} & {LB + mersec} a group.
- div is a homomorphism
- Pull back of V.b.
- Pull back of L.B. (sec)
  $\|$
  Pull back of div.
- Discuss ⊗ on tran. fun.

If time permits, revisit the hyperell. curve and its compactification.