Riemann Roch -

$$h^0(X,D) = d+1-g + h^0(K-D).$$

Cor 1) Genus 0 $\Rightarrow \mathbb{P}^1$
2) Genus 1 $\Rightarrow$ • Double cover of $\mathbb{P}^1$ branched over 4 pts
• Cubic plane curve.

3) Genus 2 $\Rightarrow$ • Double cover of $\mathbb{P}^1$ branched over 6 points. $X \to \mathbb{P}^1$ given by K_X.

Prop: Let $B \subset \mathbb{P}^1$ be a finite set of even cardinality. Then there is a unique pair (X, φ) where X is a compact R.S. and $\varphi : X \to \mathbb{P}^1$ is a 2:1 map branched over B.

So Genus 1 : $y^2 = f(x)$ cubic / quartic.

Genus 2 : $y^2 = f(x)$ quintic / sextic.

"Moduli space".

$\{$ Genus 2 curves up to iso $\}$ $\longleftrightarrow$ $\{$ Six points on $\mathbb{P}^1$ upto iso $\}$

$\updownarrow$

3 dim $\quad \| \quad \mathbb{P}^6 / PGL(2) \leftarrow \{$ Homog. sextics $\sum a_i X^i Y^{6-i}$ up to scaling $\}$

$\leftarrow$ up to changes of coordinates.

Genus 3 & above.

X. Try $X \subset \mathbb{P}^N$. Try. K_X.

$$h^0(X, K_X) = 2g-2+1-g - h^0(X,\mathcal{O})$$
$$= g.$$

$$h^0(X, K_X - p - q) = 2g-4+1-g - h^0(X, p+q).$$

What can we say about $h^0(X, p+q)$.

$$1 \le h^0(X, p+q) \le 2.$$

If $h^0(X, p+q) = 2$, then $|p+q|$ gives a degree 2 map $X \to \mathbb{P}^1$.
i.e. X is hyperelliptic.

Suppose X is not hyperelliptic.
Then
$$h^0(X, p+q) = 1.$$

so
$$h^0(X, K_X - p - q) = g-2.$$
$$= h^0(X, K_X) - 2.$$

Thm: If X is a non-hyperelliptic R.S. of $g \ge 3$, then $|K_X|$ gives an embedding

$$X \hookrightarrow \mathbb{P}^{g-1}.$$

(canonical embedding)

Genus 3: X non hyperell.

$X \hookrightarrow \mathbb{P}^2$. canonical embedding.

Homog. equations of the image?

$$r: \quad H^0(\mathbb{P}^2, \mathcal{O}(n)) \to H^0(X, \mathcal{O}(n)|_X)$$

$H^0(\mathbb{P}^2, \mathcal{O}(n)) =$ Homog. poly of degree n in X, Y, Z.

$H^0(X, \mathcal{O}(n)|_X) = H^0(X, K_X^{\otimes n})$.

$\operatorname{Ker} r =$ Homog. poly. that vanish on (the image of) X.

n	$h^0(\mathbb{P}^2, \mathcal{O}(n))$	$h^0(X, K_X^n)$
1	3	3
2	6	6
3	10	10
4	15	14

$\Rightarrow \exists$ homog. quartic F vanishing on X.
F must be irreducible.
Any other G vanishing on X is a multiple of F.
$V(F)$ must be smooth.

so $X = V(F) \subset \mathbb{P}^2$

"moduli":

$$\left\{ \begin{array}{c} \text{Non-hyp. genus 3} \\ \text{curves} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Smooth} \\ \text{Quartics in } \mathbb{P}^2 \\ / \mathrm{Aut}(\mathbb{P}^2) \end{array} \right\}$$

$$\text{"6 dim"} \leftarrow \mathbb{P}^{14} / PGL(3)$$

$$\{ \text{hyperell. genus 3} \} \longleftrightarrow \left\{ \begin{array}{c} \text{Deg 8 poly in } X, Y \\ / \mathrm{Aut}\, \mathbb{P}^1 \end{array} \right\}$$

$$\text{"5 dim"} \leftarrow \mathbb{P}^8 / PGL(2)$$

Genus 4 curves -

Hyperelliptic — $\{ \text{Deg 10 in } X, Y / PGL2 \}$

$$\mathbb{P}^{10} / PGL(2)$$

"7 dim."

Non hyperelliptic.

6x

$$X \subset \mathbb{P}^3$$

Equations?

n	$h^0(\mathbb{P}^3, \mathcal{O}(n))$	$h^0(X, K_X^n)$
1	4	4
2	10	9
3	20	15

$\rightsquigarrow Q$. quadric.
$+ C$ cubic.

Thm: $X = Q \cap C$.

"moduli" : $\longrightarrow$

$$\begin{array}{ll} Q_2 & \leftarrow 9 \text{ dim} \\ C & \leftarrow 20-4-1 = 15 \text{ dim} \\ \big/ PGL_4 & \end{array}$$

$$\begin{array}{r} 24 \text{ dim} \\ -15 \\ \hline 9 \text{ dim} \end{array}$$

Higher genus – $X \subset \mathbb{P}^{g-1}$ ... the equations become complicated.

For $g \gg 0$, X is no longer a "complete intersection."

But the idea works –

"Moduli space" of "right kind" of equations in g homog. coordinates $\big/ PGL_g$.

$\rightsquigarrow$ Hilbert Scheme

Tricky $\rightsquigarrow$ GIT.

If time permits, $|K_X|$ for X hyperelliptic.