The Canonical Embedding

- X hyperell. or $X \subset \mathbb{P}^{g-1}$ by K

Genus 2 • X hyperell.

$$\mathcal{M}_2 \cong U/_{PGL_2} \qquad U \subset \mathbb{P}^6 = \mathbb{P}\,\mathrm{Sym}^6\langle x,y\rangle$$

Genus 3 • X hyperell or

$X \subset \mathbb{P}^2$ as a plane quartic

$$\mathcal{M}_3 = U_1/_{PGL_3} \cup U_2/_{PGL_2}$$

$$U_1 \subset \mathbb{P}\mathrm{Sym}^4\langle x,y,z\rangle = \mathbb{P}^{14} \qquad U_2 \subset \mathbb{P}\mathrm{Sym}^8\langle x,y\rangle = \mathbb{P}^8$$

$= 6$ dim $\cup$ 5 dim.

Genus 4 : X hyperell or

$X \subset \mathbb{P}^3$

n	$\mathrm{sym}^n\langle 4\rangle$	$h^0(K_X^n)$	
1	4	4	
2	10	9	$\Rightarrow Q_2$
3	20	15	$\Rightarrow$ 5 cubics 1 new.

so $X \hookrightarrow Q_2 \cap Q_3$.

Turns out $X \xrightarrow{\sim} Q_2 \cap Q_3$.

and $Q_2 \cap Q_3$ is smooth.

(Possible digression – Complete intersections).

and conversely, and smooth $Q_2 \cap Q_3$ is a canonically embedded curve of genus 4.

So $\mathcal{M}_4 = \{(Q_2, Q_3 \bmod Q_2)\} / PGl_4$
$$\cup$$
$$\{Q_{10}\} / PGL_2$$

$Q_2 \leftarrow 9$ dim
$Q_3 \bmod Q_2 \leftarrow 20-4-1 = 15$ dim.

24 dim $-$ 15 $=$ 9 dim.
$$\cup$$
7 dim.

<u>Genus 5</u>: X hyperell. OR

$X \hookrightarrow \mathbb{P}^4$

n	$\mathrm{Sym}^n(5)$	$h^0(K_X^n)$
1	5	5
2	15	12

$\Rightarrow$ 3 Quadrics.

<u>Thm</u>: A canonical curve of genus 5 in $\mathbb{P}^4$ is cut out by 3 quadrics unless X admits a deg 3 map to $\mathbb{P}^1$ or a degree 5 map to $\mathbb{P}^2$.

Terminology: g^r_d = linear system of degree d and dim $(r+1)$.

More generally a canonical $X \subset \mathbb{P}^{g-1}$ is cut out by quadratic eq's unless X has a g^1_3 or a g^2_5.

(Enrique-Babbage-Petri) <u>1920's</u>.

$\boxed{|K_X|} \longrightarrow X$

- g^1_2 → Hyperell
- non hyp. → $X \subset \mathbb{P}^{g-1}$
  - g^1_3 or g^2_5 → Not cut by quadrics
  - Cut by Quadrics.
    - ?
    - ?

Green's Conjecture (1984).

"Algebra" of homog eqns of canonical X

$$\Updownarrow$$

Existence of "special" linear series on X.

One half – What does a g^1_3 or g^2_5 have to do with this?

Geometric Riemann Roch.

$$h^0(D) = d - g + 1 + \underline{\underline{h^0(K-D)}}.$$

D effective.

$$H^0(K-D) \cong \{\sigma \in H^0(K) \mid (\sigma) \geq D\}.$$

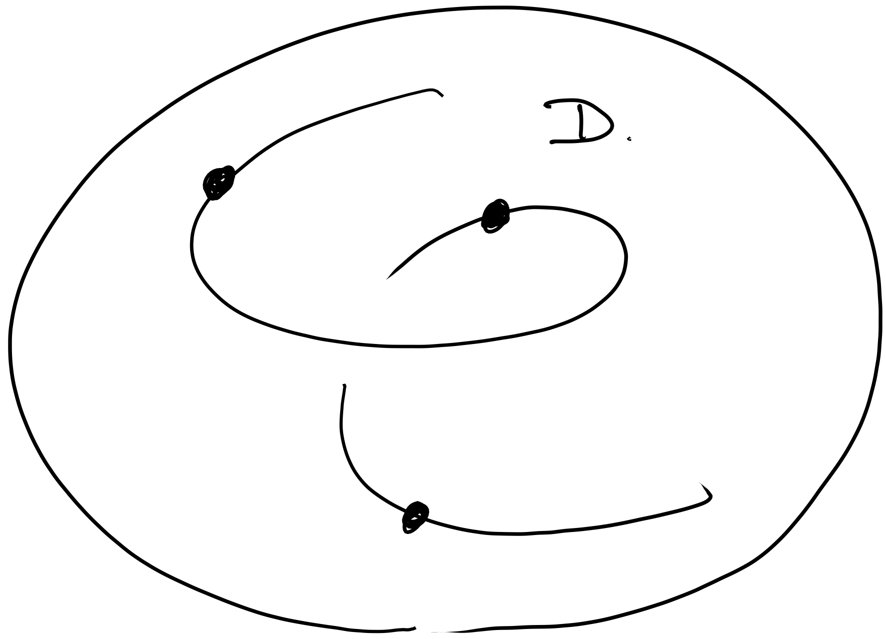
i.e. σ vanishing on D.

$X \hookrightarrow \mathbb{P}^{g-1}$. Then $H^0(X, K) = H^0(\mathbb{P}^{g-1}, \mathcal{O}(1))$

= Linear homog. pol.

$H^0(X, K-D)$ = Linear homog. pol. vanishing on D

Hyperplanes containing D.



Span(D) = linear span of D $\cong \mathbb{P}^k$

Then Lin Poly vanishing on D
$\parallel$
Lin poly vanishing on $\mathbb{P}^k$.
$\hookrightarrow$ dim $(g-k-1)$

so $h^0(X, D) = d+1-g + g-k-1$

$= d-k.$

$h^0(X, D). = \deg D - \dim \operatorname{Span}(D).$

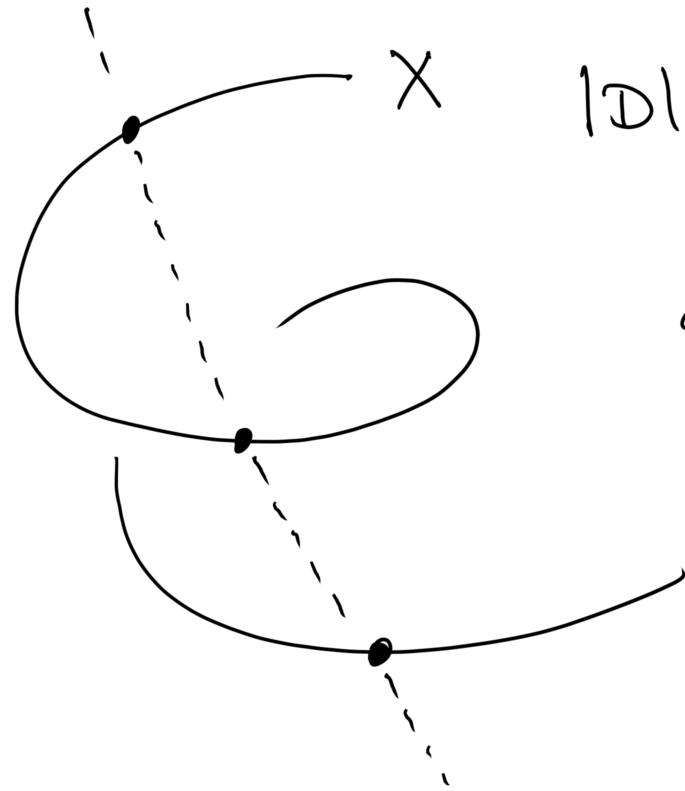
Suppose $d \leq g.$

Expect $k = d-1$

so $h^0(X, D) = 1$

More linear coincidences on D. $\Rightarrow$ higher $h^0(X, D).$

g^1_3



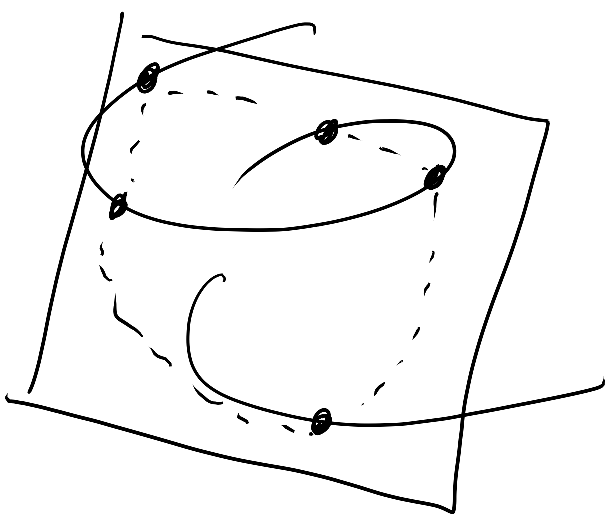
$|D|$ is g^1_3

$\Updownarrow$

$\deg D = 3$ & D lies on a line.

Any quadric containing X contains this line.
$\Rightarrow$ X can't be cut out by quadrics!

g^2_5



$\deg D = 5$ & D lies on a plane.

$\exists$ Conic through D. Any Quadric containing X must contain this conic.

$\Rightarrow$ X can't be cut out by quadrics!

So we proved the easy half of B.E.P. thm.

The canonical series for hyperell. X.

X : $y^2 - f(x) = 0$ & ......

$f(x)$ of degree $2g+2$.

$|K_X|$: $\frac{dx}{y} \cdot \langle 1, x, x^2, \dots, x^{g-1} \rangle$.

so $X \longrightarrow \mathbb{P}^{g-1}$

$(x,y) \longmapsto [1 : x : x^2 : \dots : x^{g-1}]$.

so $X \xrightarrow{2:1} \mathbb{P}^1 \underset{RNC.}{\hookrightarrow} \mathbb{P}^{g-1}$

Cor: $X \xrightarrow{2:1} \mathbb{P}^1$ is unique!