Branched Covers.

$\varphi : X \to Y$ finite map between compact RS.
$Y \supset B = \mathrm{br}\,\varphi$, the branch locus.

Then $X - \varphi^{-1}(B) \longrightarrow Y \setminus B$ is a covering space.

Thm: Given a (connected) covering space φ

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, phantom, "\cap" rotate=-90] & Y \setminus B \arrow[d, phantom, "\cap" rotate=-90] \\ X \arrow[r] & Y \end{tikzcd}$$

Then there exists a unique compact R.S. X that completes the diagram above.

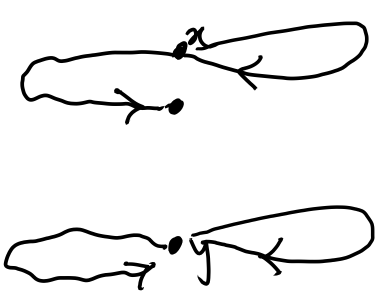
Covering spaces.

(Y, y) a connected (pointed) top. space (CW complex).
We have a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Connected pointed} \\ \text{cov. spaces of } (Y,y) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Subgroups of} \\ \pi_1(Y,y) \end{array} \right\}$$

$$\text{degree} \longleftrightarrow \text{index.}$$

$$\begin{array}{ccc} X, x & & \operatorname{im} \pi_1(X,x) \subset \pi_1(Y,y) \\ \downarrow & \rightsquigarrow & \| \\ Y, y & & \left\{ \begin{array}{l} \text{Loops in } Y \text{ based at } y, \\ \text{which lift to closed} \\ \text{loops based at } x \end{array} \right\} \end{array}$$



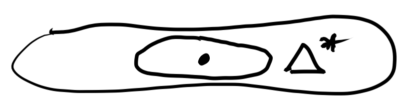
Ex: $\Delta^* =$  Punctured unit disk
$= \{ z \in \mathbb{C} \mid |z| \leq 1 \}$.

Then $\pi_1(\Delta, 1) = \mathbb{Z}$.
There is a unique subgroup of $\mathbb{Z}$ of index n.
$\Rightarrow$ There is a unique (pointed) cov. space of $(\Delta^*, 1)$ of degree n.

This one:
$$\begin{tikzcd} \tilde{\Delta}^* = \{ \omega \in \mathbb{C} \mid |\omega| \leq 1 \} \arrow[d] \\ \Delta^* \end{tikzcd}$$
$\omega \mapsto \omega^n$

Proof of thm:

Given
$$\begin{tikzcd} U \arrow[d] \\ Y \setminus B \end{tikzcd} \qquad \begin{tikzcd} \varphi^{-1}(\Delta^*) \arrow[d] \\ {} \end{tikzcd}$$



Let D be a connected component of Δ^*.
Then we have an isomorphism.

$$\begin{tikzcd} D \arrow[r, "\sim"] \arrow[d, "\varphi"'] & \tilde{\Delta}^* \arrow[d, "\pi"] \\ \Delta^* \arrow[r, equal] & \Delta^* \end{tikzcd}$$

A priori $\sim$ is only a top. iso. But both φ & π are hol. and local iso.
$\Rightarrow$ so is $\sim$.

"Plug the hole": Let $U^+ = U \cup \{p\}$
p a new point (morally $p = 0$ of $\tilde{\Delta}^*$).

Topology on U^+: A set is open iff its intersection with U and $D \cup \{p\}$ are open
where $D \cup \{p\} \longleftrightarrow \tilde{\Delta}$.

Complex chart around p given by $D \xrightarrow{\sim} \tilde{\Delta}$.

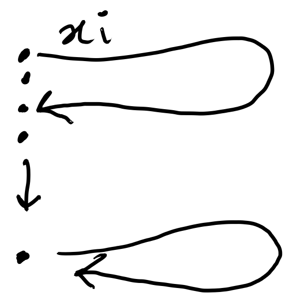
Then $U \to Y \setminus B$ extends to a hol. map

$$\begin{aligned} U^+ &\to Y \\ p &\mapsto b \end{aligned} \qquad \text{locally} \qquad \begin{aligned} \tilde{\Delta} &\longrightarrow \Delta \\ w &\longmapsto w^n. \end{aligned}$$

Repeat this for all connected components of $\varphi^{-1}(\Delta^*)$ and all $b \in B$.

<u>Monodromy</u>:

$$\begin{tikzcd} (X, x) \arrow[d, "\varphi"] \\ (Y, y) \end{tikzcd}$$



Covering space (both X, Y connected).

$S = \varphi^{-1}(y) \ni x$

We have a map

$$\mu : \pi_1(Y, y) \to \mathrm{Aut}(S).$$

$$\begin{array}{ccl} \text{loop} & \longmapsto & [x_i \mapsto x_j \\ l \text{ at } y & & \quad \text{if lift } \tilde{l} \text{ of } l \\ & & \quad \text{starting at } x_i \text{ ends at} \\ & & \quad x_j \ ]. \end{array}$$

Image of μ = Transitive subgroup of $\mathrm{Aut}(S)$.

$\mathrm{Im}\, \pi_1(X, x)$ = Stabilizer of x in $\pi_1(Y, y)$. under its action on S via μ.

If $d = \deg \varphi$ then
$S \cong \{1, 2, \dots, d\}$ (non-uniquely)

so $\mathrm{Aut}(S) \cong S_d$ (up to conjugation).

$$\left\{\begin{array}{c}\text{Connected} \\ \text{Cov. spaces of} \\ Y, y \text{ of deg } d\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Hom.} \\ \pi_1(Y,y) \to S_d \\ \text{whose image is} \\ \text{transitive}\end{array}\right\} \Big/ \text{Conjugation.}$$

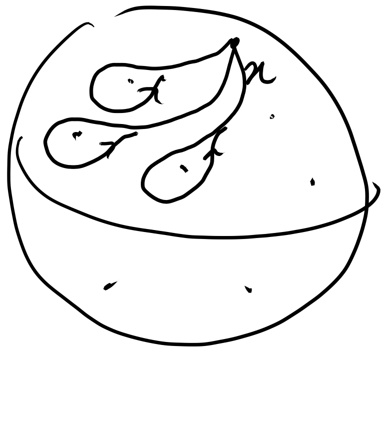
$$\left\{\begin{array}{c}\text{Conn} \\ \text{Branched covers} \\ \text{of } Y \text{ with} \\ \text{br locus} \subset B \\ \text{of deg } d\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conn. cov.} \\ \text{spaces of} \\ Y - B \\ \text{of deg } d\end{array}\right\}$$

$$\updownarrow$$

$$\left\{\begin{array}{c}\mu : \pi_1(Y \setminus B) \to S_d \\ \text{with trans. image}\end{array}\right\} \Big/ \text{Conj.}$$

<u>Double covers of $\mathbb{P}^1$ br. on B</u>

$B = \{x_1, \ldots, x_{2n}\}$.



$\pi_1(\mathbb{P}^1 \setminus B)$

$\shortparallel$

Free group on $(2n-1)$ gens.

$$= \langle \gamma_1, \dots, \gamma_{2n} \mid \gamma_1 \cdots \gamma_{2n} \rangle = 1$$

Maps to $S_2 = \mathbb{Z}/2\mathbb{Z}$ ← Abelian.

So look at the abelianization.

$$\langle \bar{\gamma}_1, \dots, \bar{\gamma}_{2n} \mid \bar{\gamma}_1 + \dots + \bar{\gamma}_{2n} = 0 \rangle$$

$$\cong \mathbb{Z}^{2n-1} \subset \mathbb{Z}^{2n}$$

How many maps? $\mathbb{Z}^{2n-1} \longrightarrow \mathbb{Z}/2\mathbb{Z}$

Many!

But we want $B = \mathrm{br}\,\varphi$

so every $\gamma_i \mapsto 1 \in \mathbb{Z}/2\mathbb{Z}$.

$\Rightarrow$ Unique!.

Gives an extremely powerful (but also extremely non-algebraic) way of constructing Riemann Surfaces.

eg. R.S. of genus 7

$$X \xrightarrow{3} \mathbb{P}^1 \qquad \mathrm{br}\ x_0, x_1, \dots, x_{18}$$

μ: $\gamma_1 \mapsto (12)$, $\vdots$, $\gamma_{16} \mapsto (12)$, $\gamma_{17} \mapsto (13)$, $\gamma_{18} \mapsto (13)$.