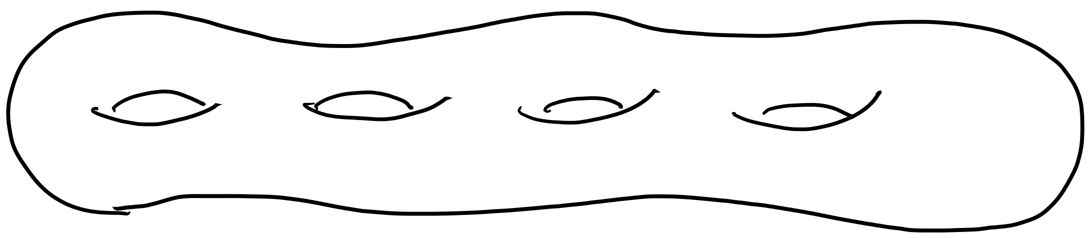
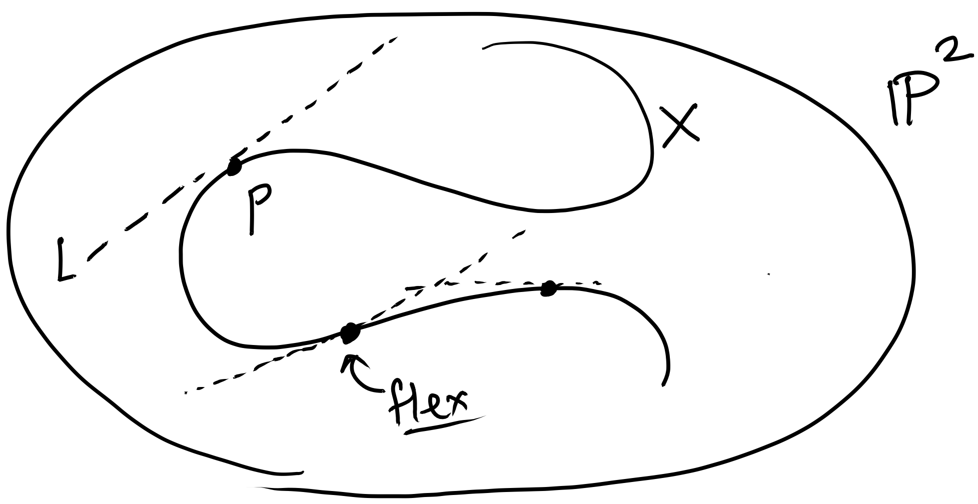
Ramification, inflection, Weierstrass points

X a compact Riemann surface.



Locally, X looks the same at all points.
But, somewhat surprisingly, the global properties of X pick out certain distinguished points on X called "Weierstrass points".

The idea behind Weierstrass points is ancient.
It is most transparent for plane curves (real picture)



Expectation: The tangent line to X at p has order of contact 2 with X at p.
(Order of contact = order of vanishing at p of the equation of L restricted to X.)

But, for some p, the order of contact may be 3 or more. Such points are called "flex points".

Weierstrass points are a special generalization of flex points.

A second look at flex points, with a different POV.

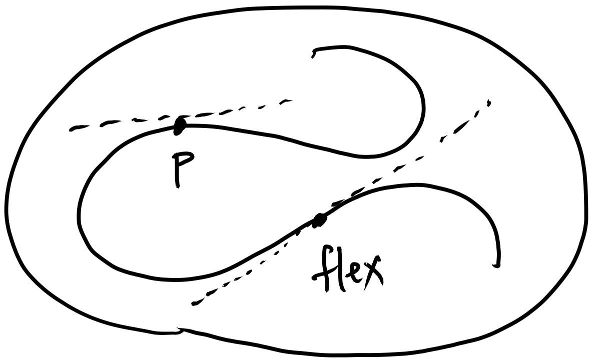
Plane curve $\rightsquigarrow (X, L, V)$ $\quad V \subset H^0(X,L)$, $\dim V = 3$

More generally, consider $\dim V = r$.

For every $p \in X$ consider the set

$$V_p = \{ \operatorname{ord}_p(\sigma) \mid \sigma \in V \}.$$

<u>Example</u>:



$$V_p = \{0, 1, 2\} \quad (p \text{ not flex})$$

$$= \{0, 1, n\} \quad n \geq 3 \quad (p \text{ flex}).$$

Prop: (X,L,V) as above. Then $\forall\, p \in X$, the set V_p contains r non-negative integers.

<u>Proof</u> - Gaussian elimination.

1) $|V_p| \leq r$.

<u>Pf</u>: Suppose $\sigma_1, \dots, \sigma_l \in V$ such that $\operatorname{ord}_p(\sigma_i)$ are <u>distinct</u>. Then $\sigma_1, \dots, \sigma_l$ must be $\mathbb{C}$ linearly independent.

$\Rightarrow \quad l \leq r$.

2) $|V_p| \geq r$.

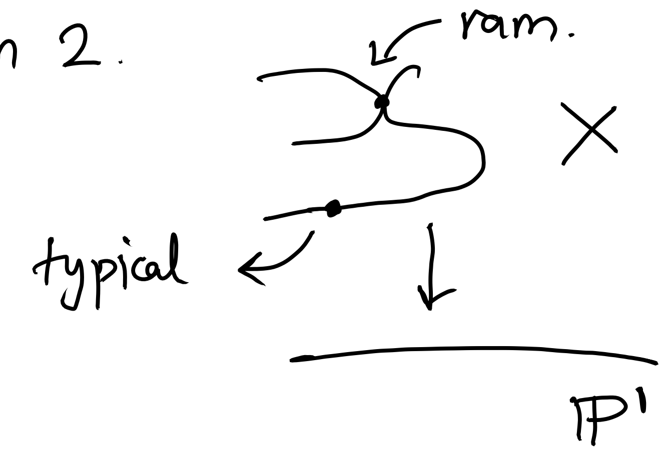
Let $\sigma_1 \in V$ have smallest ord_p, say n_1. Then $\{\sigma \in V \mid \operatorname{ord}_p \sigma > n_1\} \subset V$ is a sub space of dim $(r-1)$. Induct on r.

$\square$

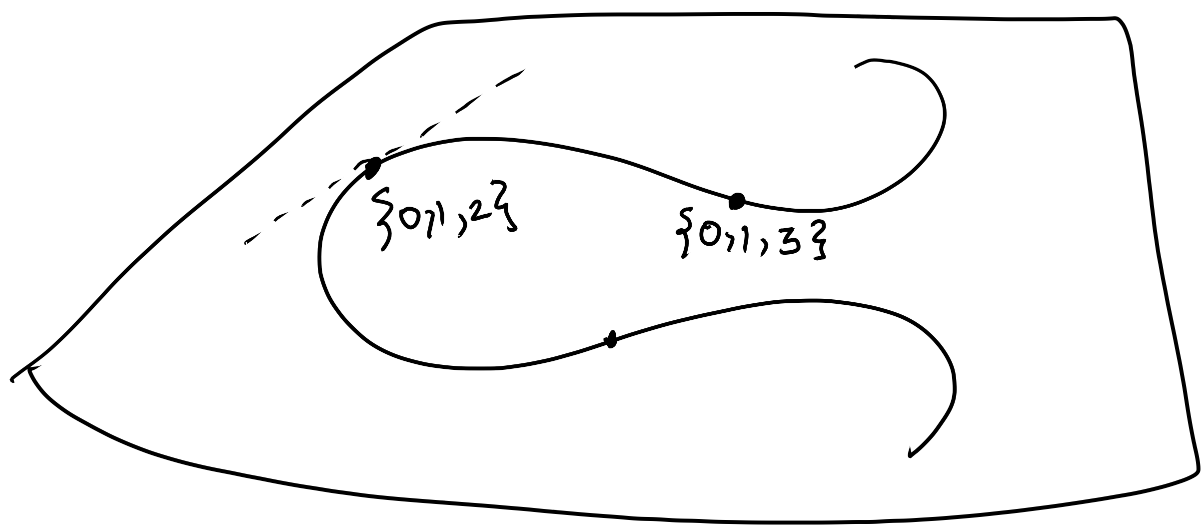
Def: We say that p is an <u>inflection / ramification point</u> of (L, V) if

$$V_p \neq \{0, 1, 2, \ldots, r-1\}.$$

Example: 1) V bpf of dim 2.



2) V bpf of dim 3



3) p is a bp of $V \iff V_p = \{1, \ldots\}$

4) V bpf + separates tangent vectors

$$\iff V_p = \{0, 1, \ldots\} \quad \forall p.$$

Prop: There are finitely many ramification points

(i.e. for all but finitely many points, the vanishing sequence is $\{0, 1, \ldots, r-1\}$).

Proof: Let $p \in X$. $\quad m_p \subset \mathcal{O}_{X,p}$ max ideal.

Consider $\quad L/m_p^r L. = L|_{rp}$

as a $\mathbb{C}$-vector space.
Explicitly if t is a uniformizer at p, then

$$L|_{rp} \underset{\substack{\downarrow \\ \text{depends on a trivialization of } L \text{ around } p.}}{\cong} \mathbb{C}[t]/_{t^r} = \mathbb{C}\langle 1, \dots, t^{r-1} \rangle.$$

We have a map

$$L \to L|_{rp}.$$

so a map $\quad H^0(X, L) \to H^0(L|_{rp})$.

explicitly $\quad \sigma \rightsquigarrow f(t) \in \mathbb{C}[[t]] \rightsquigarrow \bar{f}(t) \in \mathbb{C}[[t]]/_{t^r}$

$\downarrow$

Using chosen trivialization

So we get a map

$$V \xrightarrow{M.} H^0(L|_{rp}).$$

Claim: p is typical iff this map is an iso.

Now we compute. Let $\sigma_1, \dots, \sigma_r$ be a basis of V.
$f_1, \dots, f_r$ their local expansions. Then

$$M = \begin{pmatrix} f_1(0) & & f_r(0) \\ f_1'(0) & \ddots & f_r'(0) \\ f_1''(0) & & \vdots \\ \vdots & & f_r''(0) \end{pmatrix}.$$

Let $M_t = \begin{pmatrix} f_1(t) & \cdots & f_r(t) \\ f_1'(t) & & \vdots \\ \vdots & & \vdots \\ f^{(r-1)}(t) & & f_r^{(r-1)}(t) \end{pmatrix}$

Claim: $\det M_t \not\equiv 0$.

Pf. Wlog. $f_i(t) = t^{n_i} + \text{h.o.t.}$
with $n_1 < n_2 < n_3 < \cdots$

$$M_t = \begin{pmatrix} t^{n_1}\cdots & \cdots & & t^{n_r}\cdots \\ n_1 t^{n_1-1} & n_2 t^{n_2-1} & & n_r t^{n_r-1} \\ \vdots & & * t^{n_3-2} & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \cdots & \cdots & * t^{n_r-r+1} \end{pmatrix}$$

"Wronskian"

The lowest order term in $\det M_t$ is

$$\underbrace{\det \begin{pmatrix} 1 & 1 & 1 \\ n_1 & n_2 & n_r \\ n_1(n_1-1) & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}}_{\text{non-zero (Van der Monde)}} t^{n_1+(n_2-1)+\cdots+(n_r-r+1)}$$

$\square$

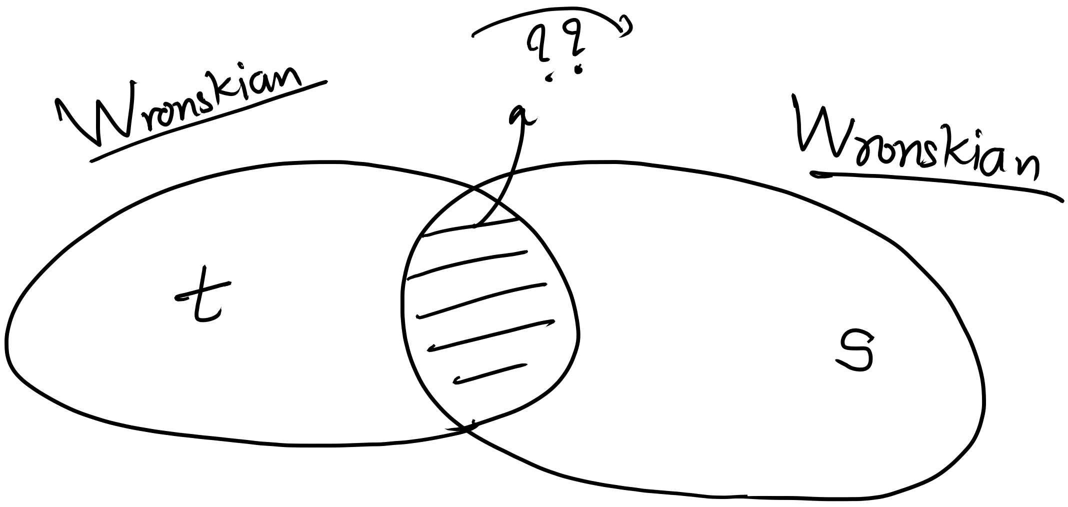
Obvious Q: How many flex points?

A: "Globalize" the Wronskian. It should be the section of a line bundle (cooked out of L and may be the cotangent / tangent bundles).

Then # Ram pts = # Zeros of Wronskian
= deg of the Wronskian's line bundle

$W_t = \det M_t$.

Which line bundle is W_t a section of ?
$\updownarrow$
What are the patching functions for W_t ?



σ (section of L)

$f \qquad\qquad\qquad\qquad f \cdot \underset{=}{\alpha} \rightsquigarrow$ trans. fun. of L

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{pmatrix} f \\ \partial f/\partial t \\ \partial^2 f/\partial t^2 \\ \vdots \end{pmatrix} \qquad\qquad \begin{pmatrix} f\alpha \\ \partial(f\alpha)/\partial s \\ \partial^2(f\alpha)/\partial s^2 \\ \vdots \end{pmatrix}$$

$$\frac{\partial (f\alpha)}{\partial s} = \alpha \frac{\partial t}{\partial s} \cdot \frac{\partial f}{\partial t} + f \frac{\partial \alpha}{\partial s}$$

$$\frac{\partial^2 (f\alpha)}{\partial s^2} = \alpha \left(\frac{\partial t}{\partial s}\right)^2 \frac{\partial^2 f}{\partial t^2} + \cdots\cdots$$

$$\frac{\partial^3 (f\alpha)}{\partial s^3} = \alpha \left(\frac{\partial t}{\partial s}\right)^3 \frac{\partial^3 f}{\partial t^2} + \cdots \leftarrow \text{lower derivatives of } f.$$

so
$$\begin{pmatrix} f\alpha \\ \partial(f\alpha)/\partial s \\ \partial^2(f\alpha)/\partial s^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha & & * & \\ & \alpha\left(\frac{\partial t}{\partial s}\right) & & \\ & & \alpha\left(\frac{\partial t}{\partial s}\right)^2 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} f \\ \partial f/\partial t \\ \vdots \end{pmatrix}$$

$$W(s) = \underbrace{\alpha(s)^r \cdot \left(\partial t/\partial s\right)^{r(r-1)/2}}_{\text{Transition function}} W(t)$$

$$1 \cdot dt = \left(\frac{\partial t}{\partial s}\right) \cdot ds$$

transition func of Ω_X

So W is a section of $L^r \otimes \Omega_S^{r(r-1)/2}$.

$$\Rightarrow \deg(W) = r \cdot \deg L + \frac{r(r-1)}{2}(2g-2)$$

Ex. 1) Plane cubic $\quad g=1, \ r=3, \ \deg L = 3$

$$3 \cdot 3 + 0 = 9$$

2) Plane quartic $\quad g=3, \ r=3, \ \deg L = 4$

$$12 + 3 \times (4) = 24$$