RIEMANN–ROCH

1. Ample divisors

Let X be a compact Riemann surface. A divisor A on X is called *ample* if for every coherent $\mathcal{O}_X$-sheaf F on X, we have

$$H^i(X, F \otimes \mathcal{O}_X(nA)) = 0$$

for $i > 0$ and for sufficiently large n.

Theorem 1.1. *There exists an ample divisor on X.*

In the complex analytic world, the proof I know of Theorem 1.1 uses harmonic analysis. The theorem (generalized naturally to all dimensions) is known as the Kodaira vanishing theorem. In the algebraic world, the proof goes by reducing the statement to a similar statement about sheaves on projective space. The theorem (generalized) is known as the Serre vanishing theorem. By GAGA, for projective algebraic varieties the statements in the analytic and the algebraic category are equivalent.

We will not prove Theorem 1.1. It is very likely that you will prove the Kodaira vanishing theorem or the Serre vanishing theorem (or both) in your mathematical life, in their natural settings. In this class, we will just reap the benefits. In the book, Miranda does something similar—he assumes the existence of enough meromorphic functions, which is a consequence of vanishing, and works with custom-defined H^1 spaces, which turn out to be the same as the standard H^1 spaces as a consequence of vanishing.

Recall that if D is a divisor and E is an effective divisor, then we have the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+E) \to \mathcal{O}_X(D+E)|_E \to 0.$$

By the long exact sequence on cohomology associated to this sequence, the following are easy to check.

1. A is ample if and only if nA is ample for some $n > 0$.
2. $H^i(X, D) = 0$ for all D and $i \geq 2$.
3. $H^i(X, D)$ are finite dimensional vector spaces (this lets us define the Euler characteristic).
4. If $H^1(X, D) = 0$ and E is effective, then $H^1(X, D+E) = 0$.
5. If A is ample and E is effective, then $A + E$ is ample.
6. If A is ample, and D is any divisor, then for all sufficiently large n, the function $n \mapsto h^0(X, D+nA)$ is a linear function of n. More precisely, we have
$$h^0(X, D+nA) = n \deg A + c$$
for some constant c and sufficiently large n.
7. If A is ample, then sufficiently large multiples nA of A separate points and tangent vectors; that is, they are very ample.

We also get some exciting information about M_X, the field of meromorphic functions on X.

Theorem 1.2. *M_X is a finitely generated field of transcendence degree 1 over $\mathbb{C}$.*

Proof. Let f be a non-constant meromorphic function on X. The function f gives a map $X \to \mathbb{P}^1$, which in turn gives an inclusion of fields

$$\mathbb{C}(t) = M_{\mathbb{P}^1} \to M_X.$$

By the next theorem, we get that $\mathbb{C}(t) \subset M_X$ is a finite extension. □

Theorem 1.3. *Let $\phi\colon X \to Y$ be a non-constant map of degree d. Then $M_Y \subset M_X$ is a field extension of degree d.*

Proof. Let $y \in Y$ be a point such that $\phi^{-1}(y) = \{x_1, \dots, x_d\}$ with $x_i \neq x_j$ if $i \neq j$. It is easy to construct meromorphic functions f_i on X for $i = 1, \dots, d$ such that f_i are holomorphic on $\{x_1, \dots, x_d\}$ and their restriction to $\{x_1, \dots, x_d\}$ gives d-linearly independent functions on $\{x_1, \dots, x_d\}$. For example, we may take f_i to not vanish at x_i and vanish at all other x_j. It is clear that $f_1, \dots, f_d$ are M_Y-linearly independent. Therefore, we have

$$\deg(M_X/M_Y) \geq d.$$

For the opposite inclusion, let $f_1, \dots, f_{d+1}$ be meromorphic functions on X. Let D be a divisor on X such that $f_i \in H^0(X, D)$; for example, take D to be the sum of the divisor of poles of all f_i. Let A be an ample divisor on Y. We have a map

$$\mathbb{C}^{d+1} \otimes H^0(Y, nA) \to H^0(X, D + n\phi^* A)$$

given by

$$e_i \otimes g \mapsto f_i \cdot \phi^* g.$$

We know that

$$\dim\left(\mathbb{C}^{d+1} \otimes H^0(Y, nA)\right) = (d+1) \cdot n \cdot \deg A + O(1),$$

and

$$\dim\left(H^0(X, D + n\phi^* A)\right) \leq d \cdot n \cdot \deg A + O(1).$$

Therefore, the dimension of the source must overtake the dimension of the target for some n, at which point, we have a non-zero kernel. Suppose $\sum e_i \otimes g_i$ lies in the kernel, where $g_i \in H^0(Y, nA) \subset M_Y$. Then we get the equation

$$\sum f_i g_i = 0,$$

which shows that $f_1, \dots, f_{d+1}$ are M_Y-linearly dependent. Therefore, we get

$$\deg(M_X/M_Y) \leq d.$$

□

2. Riemann–Roch

2.1. Riemann–Roch and Serre duality. We will elevate the Riemann–Roch formula to the following more precise statement.

Theorem 2.1. *Let X be a compact Riemann surface of genus g and D a divisor on X.*

(1) We have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \deg D.$$

(2) We have a perfect pairing

$$H^0(X, K_X - D) \otimes H^1(X, D) \to \mathbb{C}$$

given by the summation of residues.

(3) We have

$$\chi(\mathcal{O}_X) = 1 - g.$$

Of the three statements, the duality is the hardest to prove. The third statement is a numerical consequence of the formula

$$\chi(K_X) - \chi(\mathcal{O}_X) = 2g - 2$$

and the duality for $D = 0$, which implies

$$\chi(K_X) = -\chi(\mathcal{O}_X).$$

The first statement is an easy consequences of the following lemma and induction.

Lemma 2.2. *Let $D' = D + p$. Then $\chi(\mathcal{O}(D')) = \chi(\mathcal{O}(D)) + 1$.*

Proof. We have the exact sequence of sheaves

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D') \to \mathbb{C}_p \to 0.$$

Apply χ and win. □

2.2. **Residues and duality.** The duality statement in the Riemann–Roch theorem rests on the following.

Theorem 2.3 (The residue theorem). *Let ω be a meromorphic differential form on X. Then $\sum_{p \in X} \operatorname{Res}_p \omega = 0$.*

Theorem 2.3 allows us to define a pairing

$$H^1(X, D) \otimes H^0(X, K_X - D) \to \mathbb{C}$$

To understand the pairing, let us understand the vector spaces $H^0(X, K_X - D)$ and $H^1(X, D)$ in a concrete way. The first one is easy:

$$H^0(X, K_X - D) = \{\text{Meromorphic differentials } \omega \text{ such that } (\omega) - D \geq 0\}.$$

For $H^1(X, D)$, consider the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + nA) \to \mathcal{O}_X(D + nA)|_{nA} \to 0.$$

For sufficiently large n, the long exact sequence in cohomology and Serre vanishing yields

$$H^0(X, D + nA) \to H^0(X, \mathcal{O}_X(D + nA)|_{nA}) \to H^1(X, D) \to 0.$$

We will think of $H^1(X, D)$ as the quotient of $H^0(X, \mathcal{O}_X(D + nA)|_{nA})$ by $H^0(X, D + nA)$. Since the quotient is unchanged even after increasing n, it is sometimes convenient to think of $H^1(X, D)$ as the direct limit of the quotients as $n \to \infty$:

$$H^1(X, D) = \lim_{n \to \infty} H^0(X, \mathcal{O}_X(D + nA)|_{nA}) \Big/ \lim_{n \to \infty} H^0(X, \mathcal{O}_X(D + nA)).$$

Let us make the quotient description even more explicit, which will result in an interpretation of $H^1(X, D)$ in terms of "Laurent tails." Let $S = \operatorname{supp} A$ and suppose $\operatorname{supp} D \subset S$ (we can always arrange this by enlarging A if necessary). Also, assume (without loss of generality) that $A = \sum_{p\in S} p$. Given $p \in S$, let us pick a uniformizer t_p at p (and feel free to drop the subscript p if it is clear from context). Suppose $D = \sum_{p\in S} n_p \cdot p$. Then we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(D+nA)|_{nA}) &= \bigoplus_{p\in S} t^{-n_p-n}\mathbb{C}[t] \big/ t^{-n_p}\mathbb{C}[t] \\ &= \bigoplus_{p\in S} \mathbb{C}\langle t^{-n_p-1}, \ldots, t^{-n_p-n}\rangle. \end{aligned}$$

The natural restriction map

$$H^0(X, \mathcal{O}_X(D+nA)) \to H^0(X, \mathcal{O}_X(D+nA)|_{nA})$$

sends a meromorphic function f on X to its power series modulo t^{-n_p} at p. The collection of power series of f at the points p modulo t^{-n_p} is called the *Laurent tail* of f bounded by D. Thus, $H^1(X,D)$ is the space of Laurent tails bounded by D, modulo Laurent tails of meromorphic functions (holomorphic outside S). Explicitly,

$$H^1(X,D) = \bigoplus_{p\in S} \mathbb{C}\langle t^{-n_p-1}, t^{-n_p-2}, \ldots,\rangle \big/ \{\text{Laurent tails of } f \in M_X \text{ holomorphic outside } S.\}$$

Miranda takes the above as the definition of $H^1(X,D)$. Note that it is not at all obvious from this description that $H^1(X,D)$ is finite dimensional (but it is, thanks to Serre vanishing)!

We are now in a position to define the pairing. Define

$$\begin{aligned} H^0(X, K_X - D) \otimes H^1(X,D) &\to \mathbb{C} \\ \omega \otimes \tau &\mapsto \sum_{p\in S} \mathrm{Res}_p(\tau\omega). \end{aligned} \tag{1}$$

By the residue theorem, if τ is the Laurent tail of a meromorphic f (holomorphic outside S), then

$$\sum_{p\in S} \mathrm{Res}_p(\tau\omega) = 0.$$

Therefore, the pairing is well-defined.

A priori, the pairing depends on A. But it is easy to check that enlarging A does not change the pairing, and thus, the pairing is independent of A.

2.3. **Proof of Serre duality**.

Let

$$r\colon H^0(X, K_X - D) \to H^1(X,D)^\vee \tag{2}$$

be the map induced by the residue pairing, namely

$$r(\omega) = \mathrm{Res}(\omega, -)$$

Our goal is to prove that r is an isomorphism.

Proof of injectivity. Let $\omega \in H^0(X, K_X - D)$ be non-zero. Suppose $\omega = f(t_p)dt_p$ at p where

$$f(t_p) = t_p^k + \text{higher order terms}.$$

Since $(\omega) - D \geq 0$, we have $k - n_p \geq 0$.

Consider $\tau = t_p^{-k-1} \in H^1(X,D)$; this is the element in the direct sum decomposition of $H^1(X,D)$ which is t_p^{-k-1} at the summand indexed by p and 0 in the other summands. Then $\mathrm{Res}(\omega,\tau) = 1$. That is, $r(\omega)$ is non-zero. □

To prove surjectivity, we need some preparation. Let $f \in H^0(X,A)$. Then multiplication by f induces a surjective map

$$H^1(X,D) \to H^1(X,D+A).$$

In the Laurent tail description of H^1, this is literally the multiplication by the Laurent tails of f. Hence, multiplication by f induces an injective map

$$H^1(X,D+A)^\vee \to H^1(X,D).$$

We already know that we have a multiplication map

$$H^0(X,K_X - D - A) \to H^0(X,K_X - D).$$

The two multiplication maps are compatible with the residue pairing:

$$\mathrm{Res}(f\tau,\omega) = \mathrm{Res}(\tau, f\omega).$$

By taking $f = 1$, we get particular surjections

$$H^1(X,D) \to H^1(X,D+A),$$

and injections

$$H^1(X,D+A)^\vee \subset H^1(X,D)^\vee.$$

From here on, we will think of $H^1(X,D+A)^\vee$ as a subspace of $H^1(X,D)^\vee$ via this injection. This is analogous to how we think of $H^0(X,K_X-D-A)$ as a subspace of $H^0(X,K_X-D)$. More precisely, we have the following.

Lemma 2.4. *Let $\omega \in H^0(X,K_X - D)$ be such that $r(\omega) \in H^1(X,D)^\vee$ lies in $H^1(X,D+A)^\vee$. Then ω lies in $H^0(X,K_X - D - A)$.*

Lemma 2.5. *The function $n \mapsto h^1(X,D-nA)$ is linear in n for large n, with leading term $n\cdot\deg A$.*

Proof of surjectivity. Let $\lambda \in H^1(X,D)^\vee$ and $\omega \in H^0(X,K_X - D)$. Let $\lambda_1 = \lambda$ and $\lambda_2 = r(\omega)$. We have a map

$$H^0(X,nA) \otimes \mathbb{C}\langle \lambda_1,\lambda_2\rangle \to H^1(X,D-nA)^\vee$$

given by

$$f \otimes \lambda_i \mapsto f\lambda_i.$$

By considering the dimensions of the source and the target for large n, we get that the map must have a kernel. That is, there exists n and $f, g \in H^0(X,\mathcal{O}(nA))$ such that

$$f\lambda_1 = g\lambda_2.$$

Let $\eta = g/f\cdot\omega$; this lies in $H^0(X,K_X - D + mA)$ for sufficiently large m (possibly after enlarging A to account for newly acquired poles due to the zeros of f). Then $r(\eta) = \lambda$. Since $\lambda \in H^1(X,D)$, we get $\eta \in H^0(X,K_X - D)$. The proof of surjectivity is thus complete. □

2.4. **A concrete interpretation of Serre duality**. By Serre duality, the map

$$H^1(X,D) \to H^0(X,K_X - D)^\vee$$

is an isomorphism (in particular, an injection). Therefore, we get that a Laurent tail τ represents 0 in $H^1(X,D)$ if and only if its image is 0 in $H^0(X,K_X-D)^\vee$. The image of τ in $H^0(X,K_X-D)^\vee$ is the functional

$$H^0(X,K_X-D) \to \mathbb{C}$$

given by

$$\omega \mapsto \sum_{i=1}^{k} \mathrm{Res}_{p_i}(\tau\omega).$$

Therefore, τ arises from a global meromorphic function if and only if for all meromorphic forms ω in $K_X - D$, the sum of residues of $\tau\omega$ is zero. This condition is clearly necessary for τ to arise from a global meromorphic function by the residue theorem. Serre duality says that this is sufficient.

Example 2.6. Consider $X = \mathbb{P}^1$ with the standard coordinates x centered at $p = [0:1]$ and y centered at $q = [1:0]$. Let $r = [1:1]$ and $t = (x-1)$ Consider the tail

$$\tau = (2x^{-1} + 3 \text{ at } p, 1 + 3y \text{ at } q, -3t^{-1} \text{ at } r).$$

It represents an element of $H^1(\mathbb{P}^1, D)$ where $D = -1\cdot p - 2q + 0\cdot r$. But is $\tau = 0$ in $H^1(\mathbb{P}^1, D)$? That is, does there exist a rational function on $\mathbb{P}^1$ whose tails at p (up to the power x), q (up to the power y^2), and r (up to the power t^0) are as given (and with no poles elsewhere)?

Serre duality lets us find the answer. We have $h^0(\mathbb{P}^1, K_X - D) = 2$, and we can readily write down a basis of this space:

$$H^0(\mathbb{P}^1, K_X - D) = \mathbb{C}\left\langle dx, \frac{1}{x}dx \right\rangle = \mathbb{C}\left\langle \frac{-1}{y^2}dy, \frac{-1}{y}dy \right\rangle = \mathbb{C}\langle \omega_1, \omega_2 \rangle.$$

The sum of residues of $\tau\omega_1$ is $2 - 3 + 1 = 0$. The sum of residues of $\tau\omega_2$ is $3 - 3 + 0 = 0$. Therefore, there exists a rational function with the specified tails!

What is the rational function? As the writer of the example, I know the answer (because I started with the function first and then wrote its tails). The function is

$$\frac{2+x}{x(1-x)}.$$