Riemann surfaces and line bundles

X a Riemann surface.
Ω_X = The cotangent bundle.

<u>Thm</u>: If X is compact and admits a non-constant mer. function, then
$$\deg(\Omega_X) = 2g_X - 2.$$

Tautological bundle on $\mathbb{P}^n$.

Let $V \cong \mathbb{C}^{n+1}$.
$\mathbb{P}V = \{\text{Lines in } V\} \cong \{[X_0 : \cdots : X_n] \mid x_i \in \mathbb{C} \text{ not all zero}\}$

$\mathcal{L} = \{(x, v) \mid x \in \mathbb{P}V,\ v \in \text{Line defined by } x\}$.

Then $\mathcal{L} \xrightarrow{\pi} \mathbb{P}V$, and fibers are lines.
Charts on $\mathcal{L}$:
Let $U_i \subset \mathbb{P}V$ be the set where $X_i \neq 0$
Then
$$\pi^{-1}(U_i) = \{[X_0 : \cdots ; X_i : \cdots : X_n] ; v\}$$
$$\cong \left\{\left(\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i} ; v_i\right)\right\} \cong \mathbb{C}^n \times \mathbb{C}$$
Let this be a chart of $\mathcal{L}$.
Transition functions.
Let $U_{ij} = \{X_i \neq 0 \text{ and } X_j \neq 0\}$.

$$\begin{tikzcd} \pi^{-1}(U_{ij}) \arrow[r, "\sim", "\phi_i"'] \arrow[d, "\phi_j"', "\wr"] & U_{ij} \times \mathbb{C} \arrow[dl, "\phi_{ij}"] \\ U_{ij} \times \mathbb{C} & \end{tikzcd}$$

$$(x, v) \longmapsto (x, v_i)$$
$$(x, v) \longmapsto (x, v_j)$$

ϕ_{ij} = mult. by $\dfrac{X_j}{X_i}$

<u>Claim</u>: On $\mathbb{P}^1$, $\deg \mathcal{L} = -1$.

<u>Pf</u>:

$$\begin{tikzcd} \mathcal{L} \arrow[d, shift left] \\ \mathbb{P}^1 \arrow[u, shift left, bend right, "\text{mer. sec.}"'] \end{tikzcd}$$

$$\begin{tikzcd} U_2 \times \mathbb{C} & \{[x:1], (x, \boxed{1})\} \arrow[l] \\ \mathbb{P}^1 \supset U_2 = \mathbb{C}_x = \{[x:1]\} \arrow[ur, bend right, "\sigma"'] \end{tikzcd}$$

σ is a holomorphic & non-vanishing section on U_2.
On U_1.

$$U_1 = \{[1:y]\} \xrightarrow{\sigma} \{[1:y], (\boxed{\tfrac{1}{y}}, 1)\}.$$

$$\downarrow 2$$

$$U_1 \times \mathbb{C}$$

so σ has a pole of order 1 at ∞.

<u>Notation</u>:

$$\left.\begin{aligned} \mathcal{L} &= \mathcal{O}(-1). \\ \mathcal{O}(n) &= \mathcal{L}^{\otimes n}. \\ \mathcal{O}(1) &= \mathcal{L}^{-1} = \mathcal{L}^{\vee}. \end{aligned}\right\} \text{ on } \mathbb{P}^n.$$

$$\begin{tikzcd} \mathcal{O}(-1) \subset V \times \mathbb{P}^n \arrow[d] \\ \mathbb{P}^n \end{tikzcd} \qquad \text{Dually} \qquad \begin{tikzcd} V^{\vee} \times \mathbb{P}^n \arrow[r] \arrow[d] & \mathcal{O}(1) \arrow[dl] \\ \mathbb{P}^n \end{tikzcd}$$

An element $\lambda \in V^{\vee}$ gives a holomorphic global section of $\mathcal{O}(1)$.

$V = \mathbb{C}^{n+1}$. $\quad V^{\vee} \cong \mathbb{C}^{n+1} = \text{span}(i^{th} \text{ projection})$.

$X_i = i^{th}$ projection. $\rightsquigarrow$ section of $\mathcal{O}(1)$.

$X_i^d = X_i \otimes \cdots \otimes X_i \rightsquigarrow$ section of $\mathcal{O}(d)$.

So hom. poly of deg d in $X_0, \dots, X_n$ $\rightsquigarrow$ section of $\mathcal{O}(d)$.

Exercise: The map

$$\{\text{Hom. poly of deg } d \text{ in } X_0, X_1\} \to \{\text{Global. hol sec. of } \mathcal{O}_{\mathbb{P}^1}(d)\}.$$

is an iso of vector spaces.

(also for $\mathbb{P}^n$).

Given a hol. map $\varphi: X \to \mathbb{P}^n$ we get line bundles $\varphi^* \mathcal{O}(n)$ on X.

(Aside - recall how to pull back line bundles).

Set $L = \varphi^* \mathcal{O}(1)$.
We also get $(n+1)$ holomorphic sections of $\mathcal{L} \to X$, namely $\sigma_i = \varphi^*(X_i)$.

$\{\sigma_i\}$ have the prop that $\forall\ x \in X\ \exists\ i$ s.t. $\sigma_i(x) \neq 0$. $\circledast$

Conversely, given a line bundle $\mathcal{L}$ on X and $(n+1)$ sections $\sigma_0, \dots, \sigma_n$ of $\mathcal{L}$ satisfying $\circledast$ we get a hol. map
$\varphi: X \to \mathbb{P}^n$
such that $L = \varphi^* \mathcal{O}(1)$ & $\sigma_i = \varphi^*(X_i)$

$$\varphi : x \mapsto [\sigma_0(x) : \cdots : \sigma_n(x)]$$

$\circledast$: "Base-point free".
$\hookrightarrow$ $x \in X$ s.t. $\sigma_i(x) = 0\ \forall\ i$.

Guiding questions of geometry of alg curves (R.S.) —

R.S. as abstract manifolds. $\longleftrightarrow$ R.S. embedded in $\mathbb{P}^n$ (e.g. plane curves).

Line bundles & their sections.

$C \underset{i}{\subset} \mathbb{P}^2$ smooth plane curve defined by

$$F(X_0, X_1, X_2) = 0.$$

F homog. of degree d. $L = i^* \mathcal{O}(1)$.

What is $\deg(L)$?

Take a section, say $c_0 X_0 + c_1 X_1 + c_2 X_3 = \sigma$

where $c_i \in \mathbb{C}$.

Then σ will have exactly d zeros of mult. 1 on X.

so $\deg L = d$.

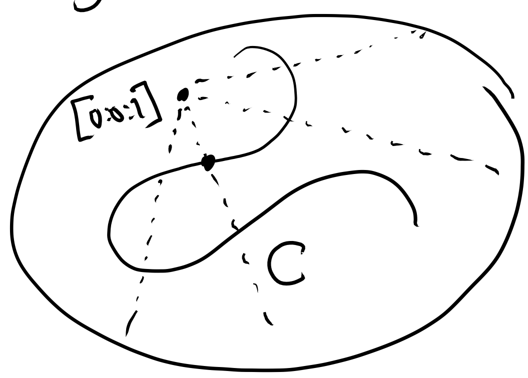
Suppose G is a homog. poly of degree e in X_0, X_1, X_2

not identically 0 on C

Then G = section of $\mathcal{O}(d)$

$\Rightarrow$ $\operatorname{div}(G)$ has degree $\underline{de}$. "Bezout's thm."

What is $g(C)$?



$\mathbb{P}^2 \quad [X:Y:Z]$

$C \xrightarrow{\varphi} \mathbb{P}^1 \quad , L = \mathcal{O}(1) \ , \ X, Y.$

Want X, Y not to vanish simult. on C

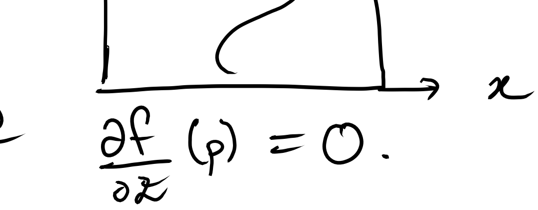
i.e. $\quad [0:0:1] \notin C.$

$$\mathbb{P}^2 - \{[0:0:1]\} \longrightarrow \mathbb{P}^1$$
$$[X:Y:Z] \longmapsto [X:Y]$$

Q: What is $\mathrm{Ram}(\varphi)$?

$$C \ni p = [A:B:C] \qquad \text{assume } B \neq 0$$
$$= [\underbrace{\tfrac{A}{B}}_{= x} : 1 : \underbrace{\tfrac{C}{B}}_{= z}] \in \mathbb{C}^2_{(x,z)}$$

$$\begin{array}{ccl} C \cap \mathbb{C}^2_{(x,z)} & \text{def. by} & f(x,z) = 0 \\ \downarrow \qquad \downarrow & & f(x,z) = F(x,1,z). \\ \mathbb{C} \qquad x. & & \end{array}$$



p is ramified iff $\frac{\partial f}{\partial z}(p) = 0.$

More: $\mathrm{Ord}_p(\mathrm{Ram}\,\varphi) = \mathrm{Ord}_p \frac{\partial f}{\partial z} = \mathrm{Ord}_p\left(\frac{\partial F}{\partial Z}\right)$

So $\mathrm{Ram}(\varphi) = \mathrm{div}\left(\frac{\partial F}{\partial Z}\right)$.

↳ section of $\mathcal{O}(d-1)$.

So $\deg \mathrm{Ram}(\varphi) = d \cdot (d-1)$

so $2g_C - 2 = (-2) \cdot d + d(d-1)$

$= d^2 - 3d$

$g_C = \frac{d^2 - 3d + 2}{2}$

$g_C = \frac{(d-1)(d-2)}{2}$

Q: 1) Are all compact R.S. of genus $\frac{(d-1)(d-2)}{2}$ plane curves of degree d?

2) What about R.S. of genus not of this form?

3) What's the "Simplest" way to exhibit a R.S. as a projective curve?