Cech Cohomology –

X a topological space.

Motivating question – Describe all the line bundles on X.

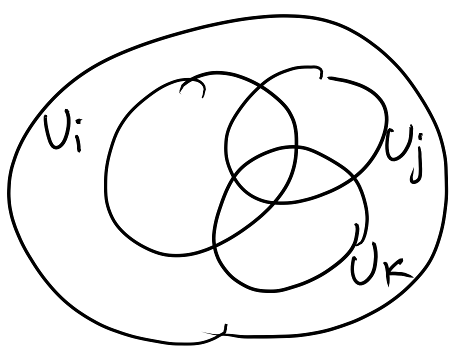
- $\pi: L \to X$. $\exists$ $\{U_i\}$ & $\phi_i : L|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}$

  Set $\phi_{ij} = \phi_i \circ \phi_j^{-1} : U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}$
  $(u,v) \mapsto (u, f_{ij}(u)\, v)$

  $f_{ij} : U_{ij} \to \mathbb{C}^*$ satisfying

  $f_{ij} \times f_{jk} = f_{ik}$. —— (*)

- Conversely, given $\{U_i\}$ and $f_{ij}: U_{ij} \to \mathbb{C}^*$ satisfying (*) , $\exists$ $\pi: L \to X$ ...



Start with $\coprod U_i \times \mathbb{C}$

$$\begin{array}{ccc} U_i \times \mathbb{C} & & U_j \times \mathbb{C} \\ \cup & \text{glue} & \cup \\ U_{ij} \times \mathbb{C} & \leftrightarrow & U_{ij} \times \mathbb{C} \\ (u,v) & \longrightarrow & (u, f_{ij}(u) \cdot v) \end{array}$$

① Two sets of $\{f_{ij}\}$ might give same (iso) L

② Only L trivialized on $\{U_i\}$ will arise in this way.

① – Suppose $\{f_{ij}\} \rightsquigarrow L$
$\{g_{ij}\} \rightsquigarrow M$

and $\exists$ iso $\eta: L \xrightarrow{\sim} M$

$$\begin{array}{ccc} L|_{U_i} & \xrightarrow{\eta} & M|_{U_i} \\ \| & & \| \\ U_i \times \mathbb{C} & \xrightarrow{\eta} & U_i \times \mathbb{C} \\ (u,v) & \mapsto & (u, h_i(u) \cdot v) \end{array}$$

$h_i : U_i \to \mathbb{C}$

$$\begin{tikzcd}
L|_{U_i} \arrow[r, "\eta"] \arrow[d, equal] & M|_{U_i} \arrow[d, equal] & L|_{U_j} \arrow[r, "\eta"] \arrow[d, equal] & M|_{U_j} \arrow[d, equal] \\
\mathbb{C}\times U_i \arrow[r, "h_i"] & \mathbb{C}\times U_i & \mathbb{C}\times U_j \arrow[r, "h_j"] & \mathbb{C}\times U_j \\
\mathbb{C}\times U_{ij} \arrow[u, phantom, "\cup" description] \arrow[r, "h_i"] \arrow[rrr, bend right=40, "f_{ij}"'] & \mathbb{C}\times U_{ij} \arrow[u, phantom, "\cup" description] \arrow[r, bend right=40, "g_{ij}"'] & \mathbb{C}\times U_{ij} \arrow[u, phantom, "\cup" description] \arrow[r, "h_j"] & \mathbb{C}\times U_{ij}
\end{tikzcd}$$

$$g_{ij} \circ h_i = h_j \circ f_{ij}$$

$$\boxed{h_i \cdot h_j^{-1} = f_{ij} \cdot g_{ij}^{-1}}$$

Conversely if $\exists \; h_i : U_i \to \mathbb{C}^*$ s.t.

$$f_{ij} \cdot g_{ij}^{-1} = h_i \cdot h_j^{-1}$$

then L & M are isomorphic

$$\boxed{\frac{\{f_{ij}\} \text{ satisfying } f_{ij} \cdot f_{jk} \cdot f_{ik}^{-1} = 0}{\{f_{ij}\} \text{ of the form } f_{ij} = h_i \cdot h_j^{-1}}} \longleftrightarrow \boxed{\begin{array}{c}\text{iso classes of} \\ \text{Line bundles} \\ \text{trivialized} \\ \text{on } \{U_i\}\end{array}}$$

$$\frac{\operatorname{Ker}(d : C^1 \to C^2)}{\operatorname{Im}(d : C^0 \to C^1)}$$

X a topological space.
F a sheaf of abelian groups on X.

<u>Examples.</u>

① X a R.S. : $F = \mathcal{O}_X$

i.e. $F(U) = \{\text{hol. func. } U \to \mathbb{C}\}$

② $F = \mathcal{O}_X^*$

$F(U) = \{\text{hol. func. } U \to \mathbb{C}^*\}$

③ $\pi : V \to X$ a v.b. $F = \mathcal{O}(V)$

$F(U) = \{\text{hol. sec. } U \to V|_U\}$.

⑤ X manifold. $F = C_X^\infty$

$V \to X$ v.b. $F = C_X^\infty(V)$

⑥ <u>Constant sheaves</u> : G an abelian group

$\underline{G}_X(U) = \{\text{Locally const. fun } U \to G\}$.

e.g. $\underline{\mathbb{Z}}_X$, $\underline{\mathbb{R}}_X$, $\underline{\mathbb{Q}}_X$, etc.

Fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X.
Given $i_0, \dots, i_n \in I$

$$U_{i_0, \dots, i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$$

Cech complex

$$C^n(U,F) = \prod_{(i_0,\dots,i_n)} F(U_{i_0,\dots,i_n})$$

map $d : C^n(U,F) \to C^{n+1}(U,F)$

$$\phi \mapsto d\phi$$

$$(d\phi)_{i_0,\dots,i_{n+1}} \in F(U_{i_0,\dots,i_{n+1}})$$

$$\parallel$$

$$\phi_{i_1,\dots,i_n} - \phi_{i_0 i_2 \dots i_n} + \phi_{i_0 i_1 i_3 \dots i_n} \cdots$$

e.g. $d : C^0 \to C^1$

$$\{\phi_i\} \mapsto \{d\phi_{ij}\}$$

$$d\phi_{ij} = \phi_i - \phi_j.$$

$d : C^1 \to C^2$

$$\{\phi_{ij}\} \to \{d\phi_{ijk}\}$$

$$d\phi_{ijk} = \phi_{jk} - \phi_{ik} + \phi_{ij}$$

Check $d \circ d = 0$.

Def: $H^i_{\mathcal{U}}(X,F) := \dfrac{\operatorname{Ker}(d: C^i \to C^{i+1})}{\operatorname{Im}(d: C^{i-1} \to C^i)}$

Rem: 1) $H^1_{\mathcal{U}}(X, \mathcal{O}_X^*)$ = Hol. Line bundles trivi ized on $\mathcal{U}$

2) $H^0_{\mathcal{U}}(X, \mathcal{O}_X) = (X, \mathcal{O}_X)$

$H^0_{\mathcal{U}}(X, \mathcal{O}_X(V) = \Gamma(X, V)$

$H^0_{\mathcal{U}}(X, F = \Gamma(X, F) = F(X).$

3) A map $F \to G$ induces a map

$$C^i_{\mathcal{U}}(F) \to C^i_{\mathcal{U}}(G)$$

commut' with d and hence a map

$$H^i_{\mathcal{U}}(X,F) \to H^i_{\mathcal{U}}(X,G)$$

Refinements:- Suppose we have open covers $\mathcal{U} = \{U_i\}_{i \in I}$ and $V = \{V_j\}_{j \in J}$. A refinement

$\mathcal{U} \to V$ is a map $r: J \to I$ such that

$$V_j \subset U_{r(j)}$$

- A refinement $\mathcal{U} \to V$ gives maps

$$C^i_{\mathcal{U}} \to C^i_V$$

Commuting with d's.

Now, open coverings under refinements forms a "directed system". → any two have common refinement.

$$\begin{tikzcd} U \arrow[dr] & \\ & W \\ V \arrow[ur] & \end{tikzcd}$$

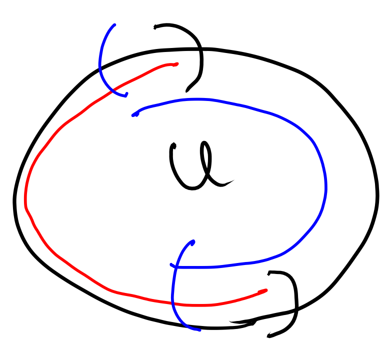
$$H^i(X,F) := \varinjlim H^i_{\mathcal{U}}(X,F)$$

So an elem. of $H^i(X,F)$ is rep by a Čech cocycle on <u>some</u> open cover $\{U_i\}$.

Two cocycles on $\{U_i\}$ are eqv. if $\exists$ refinement after which their difference becomes a coboundary.

<u>Easy calculations</u> —

<u>$\mathbb{Z}$</u> on



S^1

How

SES → LES

$$0 \to K \to M \to N \to 0$$

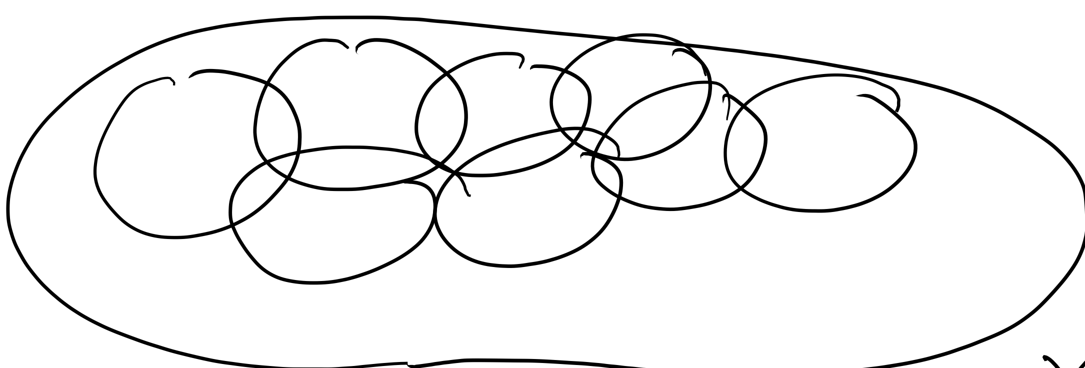
an exact seq. of sheaves of abelian groups on X. Then we get ....

$$0 \to H^0(K) \to H^0(M) \to H^0(N)$$

$$\to H^1(K) \to H^1(M) \to H^1(N)$$

$$\to H^2(K)$$

$n \in N$



X

$\exists \{U_i\}$ & $m_i \in M(U_i)$ s.t. $m_i \mapsto n|_{U_i}$

Consider $k_{ij} = m_i - m_j|_{U_{ij}} \in K(U_{ij})$

define $\delta(n) = \langle k_{ij} \rangle$.

suppose $\delta(n) = 0$. so $\exists\ k_i \in K(U_i)$

s.t. $k_{ij} = k_i - k_j$.

Set $m_i' = m_i + k_i$. Then $m_i - m_j = 0$

so $m_i = m|_{U_i}$ for some $m \in M(X)$ & $m \mapsto n$.