X a topological space
F a sheaf of abelian groups on X $\rightsquigarrow$ $H^n(X, F)$.

SES → LES

Given $0 \to K \to M \to N \to 0$ exact seq of sheaves on X.

We get

$$0 \to H^0(K) \to H^0(M) \to H^0(N)$$
$$\hookrightarrow H^1(K) \to H^1(M) \to H^1(N)$$
$$\hookrightarrow \cdots$$

Construction of the boundary homomorphism $H^0(N) \to H^1(K)$
Given $n \in H^0(X, N)$, $\exists$ open cover $\{U_i\}$ and $m_i \in M(U_i)$ such that

$$m_i \mapsto n|_{U_i}$$

Consider

$$K_{ij} = m_i - m_j \in K(U_{ij})$$

Then K_{ij} defines a 1-cocycle of K.
The boundary hom. is given by

$$n \mapsto (K_{ij})$$

- Connections with de Rham / Singular cohomology
- Vanishing theorems
- Back to divisors and line bundles.

Def: A sheaf G is called <u>acyclic</u> :-

$$H^i(X, G) = 0 \quad \forall\, i \geq 1.$$

<u>Acyclic resolutions:</u> Suppose F is a sheaf &

$$0 \to F \to G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} G_2 \xrightarrow{\cdots} \cdots$$

is an exact seq. of sheaves where G_i are acyclic. Then

$$H^n(F) = H^n[\, H^0(G_0) \to H^0(G_1) \to \cdots \to \cdots]$$

<u>Pf</u>:
$$0 \to F \to G_0 \to \underset{\substack{\| \\ F_1}}{\operatorname{Im} d_0} \to 0 \qquad \text{— } (*)$$

$$\Rightarrow H^0(F) = \operatorname{Ker}\left(H^0(G_0) \to H^0(F_1)\right)$$

but $F_1 \subset G_1$ so
$$H^0(F_1) \subset H^0(G_1).$$

Hence $H^0(F) = \operatorname{Ker}\left(H^0(G_0) \to H^0(G_1)\right).$

Also $0 \to F_1 \to G_1 \to G_2 \to \cdots$

We have the LES:

$$0 \to H^0(F) \to H^0(G_0) \to H^0(F_1)$$
$$H^1(F) \to \cancel{H^1(G_0)} = 0 \to H^1(F_1)$$
$$H^2(F) \to \cdots \to 0$$

So
$$H^1(F) = \operatorname{coker}\left(H^0(G_0) \to H^0(F_1)\right)$$
$$= \frac{\operatorname{Ker}\left(H^0(G_1) \to H^0(G_2)\right)}{\operatorname{Im}\left(H^0(G_0) \to H^0(G_1)\right)}$$

Also from LES:

$$H^i(F) = H^{i-1}(F_1) \quad \forall\ i \geq 2$$

Note: $0 \to F_0 \to G_0 \to G_1 \to \cdots$

$0 \to F_1 \to G_1 \to G_2 \to \cdots$

So we get $H^n(F) = H^n(H^0(G_0) \to H^1(G_1) \to \cdots)$ by induction on n.

$\square$

de Rham Cohomology

X a real manifold of dim n.

C^∞_X = Sheaf of C^∞ functions on X

$C^\infty(\Omega_X)$ = Sheaf of C^∞ 1-forms on X

$$\sum f_i(x_1, \ldots, x_n)\, dx_i$$
$$\hookrightarrow C^\infty$$

$C^\infty(\Lambda^2 \Omega_X)$ = Sheaf of C^∞ 2-forms on X

$$\sum f_{ij}\, dx_i \wedge dx_j$$

$\vdots$

$C^\infty(\Lambda^n \Omega_X)$ = sheaf of C^∞ n-forms.

We have a map $d: C^\infty(\Lambda^i \Omega_X) \to C^\infty(\Lambda^{i+1} \Omega_X)$

$$f\, dx_{a_1} \wedge \cdots \wedge dx_{a_i} \mapsto \sum_b \frac{\partial f}{\partial x_b}\, dx_b \wedge dx_{a_1} \wedge \cdots \wedge dx_{a_i}$$

$$0 \to \underline{\mathbb{R}} \to C^\infty_X \xrightarrow{d} C^\infty_X(\Omega_X) \xrightarrow{d} C^\infty_X(\Lambda^2\Omega_X) \to \cdots$$

This is an exact sequence of sheaves on X

Claim: $C^\infty_X(\Lambda^i\Omega_X)$ are all acyclic.

$\Rightarrow$ $H^i_{cech}(X, \underline{\mathbb{R}}) = H^i$ of

$$\underbrace{H^0(C^\infty_X) \to H^0(C^\infty_X(\Omega_X)) \to \cdots\cdots}_{H^i_{dR}(X, \mathbb{R}).}$$

Thus, for manifolds, Cech = de Rham.

Pf of Claim: Let's prove it for $F = C^\infty_X$.

Given $\{U_i\}$ and $f_{ij} \in F(U_{ij})$, $\partial f_{ij} = 0$.
Want to show its a boundary.

Let $\{\lambda_i\}: X \to \mathbb{R}$ be a partition of unity subordinate to $\{U_i\}$.

Let $g_j = \sum_k \lambda_j \cdot f_{jk} \quad \in$

Then $g_j - g_i = \sum_k \lambda_k \underbrace{(f_{jk} - f_{ik})}_{f_{ij}}$

$= f_{ij}.$

$\square$.

Same proof:-
Let F be a sheaf of C^∞_X-modules.
Then F is acyclic
e.g. $C^\infty_X(\Lambda^i\Omega_X)$ or $C^\infty_X(V)$.

Also turns out, for any ab gp R, X locally contractible

$$\check{H}^i(X, \underline{R}) = H^i_{sing}(X, R)$$

Pf idea: Construct C^i = sheaf of singular R-cochains

Get a resolution

$$0 \to \underline{R} \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{c^2} \cdots\cdots \to$$

Show C^i are acyclic. Then

$$\check{H}^i(X, \underline{R}) = H^i_{sing}(X, R).$$

$\square$

Cor: $H^i_{sing}(X, \mathbb{R}) = H^i_{dR}(X, \mathbb{R})$ for X a manifold.

Which open cover should I take?

F a sheaf on X, $U = \{U_i\}$ an open cover such that

$F|_{U_I}$ is an acyclic sheaf on U_I $\forall I$.

Then $H^i(X, F) = H^i_U(X, F)$.

Pf: Prelim - $i: V \subset X$ & G a sheaf on V.
We get a sheaf $i_* G$ "extension by 0"
on X:

$$i_*(G)(U) = G(U \cap V).$$

"Sheafify Čech coh"

Given $\{U_i\}$. Consider

$$\begin{aligned}
&F \\
&\downarrow \\
&\mathcal{G}_0 = \prod i_*(F|_{U_i}) \\
&\mathcal{G}_1 = \prod i_*(F|_{U_{ij}}). \\
&\vdots
\end{aligned}$$

∂

∂

i.e. $\mathcal{G}_i(V) = C^i(V, F|_V, \{U_i \cap V\})$

Claim: This is an exact seq. of sheaves.

Pf: Given $\sigma \in \mathcal{G}_i$ around p with $\partial\sigma = 0$.

Want $\tilde{\sigma}_{a_1,\dots,a_i}$. Have $\sigma_{a_1,\dots,a_{i+1}}$. Set

$$\tilde{\sigma}_{a_1,\dots,a_i} = \sigma_{a_1,\dots,a_i,j}.$$

Then $$(\partial\tilde{\sigma})_{a_1,\dots,a_{i+1}} = \sum \tilde{\sigma}_{a_1,\hat{a}_k,a_{i+1}} (-1)^k$$
$$= \sum \sigma_{a_1,\hat{a}_k,a_{i+1},j} (-1)^k$$
$$= \sigma_{a_1,\dots,a_{i+1}}.$$

$\square$

By construction $\mathcal{G}_i$ are acyclic.

So claim follows.

Vanishing Theorems :- X a complex manifold.

Recall, given a v.b. $V \to X$, we have a sheaf $\mathcal{O}_X(V)$ = Sheaf of hol. sections of V.

$$H^i(X, \mathcal{O}_X(V)) =: H^i(X, V).$$

① dim vanishing:

$$H^i(X, V) = 0 \quad \text{for } i > \dim_{\mathbb{C}} V$$

② Finite dim: X compact.

$$\Rightarrow \quad H^i(X,V) \text{ is a fin dim } \mathbb{C}\text{- v.space.}$$

③ Serre / Kodaira Vanishing

X projective $\quad (X \subset \mathbb{P}^N)$

$L = \mathcal{O}(1)|_X$.

Serre: $H^i(X, V \otimes L^n) = 0 \quad \forall \; i > 0$ & sufficiently large n.

Kodaira: $H^q(X, \Lambda^p \Omega \otimes L) = 0 \quad$ if $p+q > \dim_{\mathbb{C}} X$

in particular

$$H^i(X, K_X \otimes L) = 0 \quad \forall \; i > 0.$$