$$\left\{\begin{array}{c}\text{Invertible}\\ \text{Sheaves}\end{array}\right\} \longleftarrow \left\{\text{Line bundles}\right\}$$

$$\mathcal{O}_X(L) \longleftarrow\!\!\shortmid L$$

$$F \longmapsto L$$

Choose an open cover $\{U_i\}$ and iso.

$$F|_{U_i} \xrightarrow[\sim]{\sigma_i} \mathcal{O}_{U_i} \qquad F|_{U_j} \xrightarrow[\sim]{\sigma_j} \mathcal{O}_{U_j}$$

$$F|_{U_{ij}} \overset{\sigma_i}{\underset{\sigma_j}{\rightrightarrows}} \mathcal{O}_{U_{ij}}$$

$$\sigma_i \cdot \sigma_j^{-1} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}} \quad \text{multiplication by } t_{ij}$$

$$\sigma_i = t_{ij} \cdot \sigma_j .$$

Let L be the line bundle with transition functions t_{ij}

$$L|_{U_i} \xrightarrow{s_i} \mathbb{C} \times U_i \qquad L|_{U_j} \xrightarrow{s_j} \mathbb{C} \times U_j$$

$$L|_{U_{ij}} \overset{s_i}{\underset{s_j}{\rightrightarrows}} \mathbb{C} \times U_{ij}$$

$$s_i \cdot s_j^{-1} = \text{mult. by } t_{ij}.$$

Then $\mathcal{O}_X(L) \cong F.$

$$\{\text{Inv sheaves}\} = \{\text{Line bundles}\}$$
$$\begin{tikzcd} \{\text{Inv sheaves}\} \arrow[r, equal] \arrow[dr, equal] & \{\text{Line bundles}\} \arrow[d, equal] \\ & H^1(X, \mathcal{O}_X^*) \end{tikzcd}$$

Divisors.

$$\begin{tikzcd} \mathcal{O}_X(D) & \{\text{Inv sh}\} \arrow[rr, equal] & & \{\text{Line bundles}\} \\ D \arrow[u, mapsto] & & \{\text{Divisors}\}/_{\text{lin. equiv.}} \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

$D_1 \sim D_2 \Rightarrow \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$

Conversely $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \Rightarrow D_1 \sim D_2$

Pf: Let $U = X - D_1 - D_2$

Then

$$\begin{array}{ccc} 1 \in \Gamma(U, \mathcal{O}_X(D_1)) & & f^{-1} \\ \downarrow & & \uparrow \\ f \in \Gamma(U, \mathcal{O}_X(D_2)) & & 1 \end{array}$$

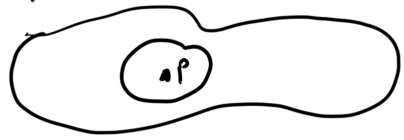
Then f is a mer. fun on X and gives the lin equiv. $D_1 \sim D_2$

$\square$

$$D \longrightarrow L$$
$$np$$

$U_1 = X - D$

$U_2 =$ small disk around p.



$t =$ uniformizer at p

$$\begin{array}{ccc} U_1 \times \mathbb{C} & & U_2 \times \mathbb{C} \\ \cup & & \cup \\ U_{12} \times \mathbb{C} & \xrightarrow{t^n} & U_{12} \times \mathbb{C} \end{array}$$

$U_1 \to U_1 \times \mathbb{C}$

$u \mapsto (u, 1)$;

We have a section

$$1_{U_1} : U_1 \to U_1 \times \mathbb{C}$$
$$u \mapsto (u, 1)$$

which on U_2 is

$$U_2 \to U_2 \times \mathbb{C}$$
$$u \mapsto (t^n, u).$$

i.e. a meromorphic section σ of L with $(\sigma) = np$.

$$\left\{ \begin{array}{l} \text{inv. sheaves} \\ \text{that admit} \\ \text{mer. sec.} \end{array} \right\} \;=\!=\; \left\{ \begin{array}{l} \text{Line bundles that} \\ \text{admit mer. sect} \end{array} \right\}$$

$\mathcal{O}(D)$ $\qquad\qquad$ L

$$\{\text{Divisor classes}\}$$

$$D$$

$$H^0(X, \mathcal{O}(D)) = \{ f \mid (f) + D \geq 0 \} \qquad f$$
$$\| \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$H^0(X, L) = \{\text{Global hol. sect. of } L\} \qquad f\sigma$$

where σ is the mer. sec. of L given by D.

$(f) + D$ is an <u>effective</u> divisor eqv to D.

$\leftarrow$ modulo scaling by inv. fun. on X.

$$|D| = \{ \text{Eff div lin eqv to } D \}$$

Thm: X a compact R.S., D a divisor on X.

Then $H^0(X, \mathcal{O}(D))$ is a fin dim. v. space.

Pf: Let $p \in X$, let t be a uniformizer.

$$L_n(p) = \mathbb{C}\langle t^{-1}, t^{-2}, \ldots, t^{-n} \rangle$$
$$= t^{-n}\mathcal{O}_{X,p} / \mathcal{O}_{X,p}$$
$$= m_p^{-n}\mathcal{O}_{X,p} / \mathcal{O}_{X,p}.$$

Let $L(D) = \prod_{p \in X} L_{\mathrm{mult}_p(D)}(p)$

↳ fin. dim. v. space.

We have a map

$$\begin{aligned} H^0(X, \mathcal{O}(D)) &\longrightarrow L(D) \quad \leftarrow \text{fin dim.} \\ f &\longmapsto \text{Laurent tails of } f \\ &\quad = \text{image of } f \text{ in the quotient.} \end{aligned}$$

Kernel = Global Hol. func. on X.

↳ $\mathbb{C}$

$\Rightarrow H^0(X, \mathcal{O}(D))$ is fin dim.

$\square$

Def: A <u>linear system</u> (of divisors equivalent to D) is a <u>subspace</u> of $H^0(X, \mathcal{O}(D))$.

$p \in X$ is a "<u>base point</u>" of the linear system if all sections in the linear system vanish at p.

Recall: A base point free linear system of dim $(n+1)$ + (choice of basis)

$$\uparrow \downarrow$$

A hol. map $X \to \mathbb{P}^n$.

Q: When is this map an embedding?

Def: Embedding: $i : X \to P$

1. i homeomorphism onto its image
2. $\forall\ x \in X$ and the germ of a hol. fun at x is the restriction of the germ of a hol. fun on Y.

i.e. the restriction map

$$\mathcal{O}_P \to i_* \mathcal{O}_X$$

is surj.

For X = Riemann surface, $\mathcal{O}_{X,p}$ = Conv. pow. ser in a uniformizer t.

so surj $\Leftrightarrow$ $\exists$ fun on P hol. at x that restricts to a uniformizer.

X compact: ① $\Leftarrow$ one-one.

X a compact R.S. D a divisor
$V \subset H^0(X, D)$ a bpf linear series.

$\varphi : X \to \mathbb{P}^n$ the map given by V.
$\quad\quad\quad\quad \| $
$\quad\quad \{[X_0 : \cdots : X_n]\}$.

Then $V = \{ \varphi^*(h) \mid h \in \mathbb{C}\langle X_0, \ldots, X_n \rangle \}$.

1) φ is 1-1 iff V "separates points".
(i.e. $\forall\, p \neq q \in X$ $\exists\, \sigma \in V$ such that $\sigma(p) = 0$ but $\sigma(q) \neq 0$).

(Pf: Pick $\sigma = \varphi^*(h)$ where h is the equation of a hyperplane through p but not q)

2) $\mathcal{O}_{\mathbb{P}^n} \to \varphi_* \mathcal{O}_X$ is surj iff V separates tangent vectors i.e.
$\forall\, p \in X$ $\exists\, \sigma \in V$ s.t. σ vanishes to order exactly 1 at p.

Pf: $p \mapsto [\sigma_0(p) : \cdots : \sigma_n(p)]$ $\quad$ $\sigma_0(p) \neq 0$, $\sigma_i(p) = 0$ $\forall\, i > 0$

So, locally at p. $X \to \mathbb{A}^n$
$$x \mapsto \left(\frac{\sigma_1}{\sigma_0}, \ldots, \frac{\sigma_n}{\sigma_0} \right).$$
$$p \mapsto (0, \ldots, 0).$$
hol. fun. at 0 in $\mathbb{A}^n$ = power series in $x_1, \ldots, x_n$.
So if $\exists$ hol fun that restricts to a uniformizer, then $\exists$ lin comb of $x_1, \ldots, x_n$ that does. The same lin. comb gives σ.

$\square$