Linear series

X a compact Riemann surface.
D a divisor on X.

$H^0(X, \mathcal{O}(D))$ is a fin dim $\mathbb{C}$ - vector space.
In fact, let D be effective, then

$$\dim H^0(X, \mathcal{O}(D)) \leq \deg D + 1.$$

(In general we get $\leq \deg(D^+) + 1$)

$$H^0(X, \mathcal{O}(D)) = \{ f \in \mathcal{M}_X \mid (f) + D \geq 0 \}$$
$$= \{ \text{Hol. sec. of corresponding } L \}.$$

$$\mathbb{P}H^0(X, \mathcal{O}(D)) = \{ E \mid E \sim D \ \& \ E \geq 0 \} =: |D|$$

Def: • A linear system is a pair (V, D) where $V \subset H^0(X, \mathcal{O}(D))$. is a vector subspace.

• Base point.

Ex: $X \xrightarrow{i} \mathbb{P}^n$. $\quad L = \varphi^* \mathcal{O}(1)$
$\quad D = \varphi^* H$

Get a linear series $V \subset H^0(X, L)$
given by $V = \{ \varphi^*(\Sigma a_i X_i) \}$

$|V| \subset |D|$ $\quad$ base-point free
$\|$
$\{$ Restrictions of hyperplanes $\}$.

$\deg(V, D) := \deg D$
$\dim(V, D) := \dim V$.

A bpf lin sys of dim $(n+1)$ gives $i : X \to \mathbb{P}^n$.

When is i an embedding?

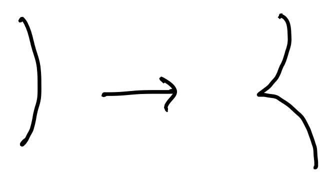
What is an embedding? X, Y complex manifolds. $\varphi : X \to Y$ is an embedding if

(1) $X \longrightarrow \varphi(X)$ is a homeomorphism.
(2) $\forall\ p \in X$ and $f \in \mathcal{O}_{X,p}$ $\exists$ $g \in \mathcal{O}_{Y,p}$ such that $f = g|_X$.

Restatement of (2): The restriction map
$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$
is a surjection.

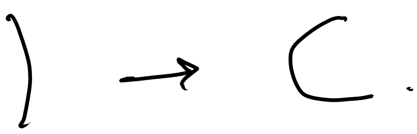
Non-Example $\Delta =$ disk $\{ z \in \mathbb{C} \mid |z| < 1 \}$

$$\varphi : \Delta \to \mathbb{C}^2$$
$$z \mapsto (z^2, z^3).$$



Then φ satisfies (1). but not (2).

Example $z \mapsto (z, z^2)$.



For a Riemann surface X, for (2), it suffices to have $g \in \mathcal{O}_{Y,p}$ s.t. $g|_X$ is a uniformizer at p.

Let us examine (2) for $X \to \mathbb{P}^n$ given by (V, L). Let $p \in X$.

$$\varphi(p) = [\sigma_0(p) : \cdots : \sigma_n(p)].$$
By a linear change of coordinates $\sigma_0(x) \neq 0$ & $\sigma_1(p), \ldots, \sigma_n(p) = 0$.

So locally near p: $\varphi(x) = \left(\frac{\sigma_1}{\sigma_0}(x), \ldots, \frac{\sigma_n}{\sigma_0}(x) \right) \in \mathbb{A}^n$
$$p \mapsto 0$$

$\mathcal{O}_{\mathbb{A}^n,0}$ = (conv) powerseries.

$\exists\ f \in \mathcal{O}_{\mathbb{A}^n,0}$ s.t. $f|_X$ is a uniformizer at p

$\Leftrightarrow \exists\ i$ s.t. $\frac{\sigma_i}{\sigma_0}(x)$ is a uniformizer at p

$\Leftrightarrow \exists\ i$ s.t. σ_i has a simple zero at p

$\Leftrightarrow \exists\ \sigma \in V$ that has a simple zero at p.

<u>Def</u>: (V, D) a linear system. We say it <u>separates tangent vectors at p</u> if $\exists\ \sigma \in V$ s.t. σ has a simple zero at p.

<u>Examine</u> (1): X compact
$X \to \mathbb{P}^n$ homeo onto image iff one-one.
iff $\forall\ p \neq q$, $\exists\ \sigma \in V$ s.t. $\sigma(p) = 0$ & $\sigma(q) \neq 0$.

<u>Def</u>: (V, D) separates points if —"—.

<u>Thm</u>: Let (V, D) be a base point free linear system. Then the induced map $\varphi: X \to \mathbb{P}^n$ is an embedding iff it separates points and tangent vectors.

Explain the terminology.

<u>Extended Example</u>: X = compactification of
$\{ z^2 = x^3 + 1 \}$
$\{ w^2 = y + y^4 \}$

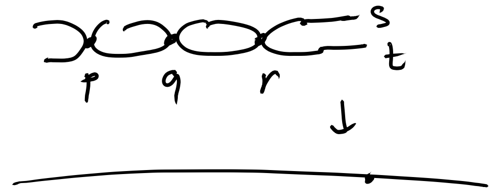
$D = 3 \cdot \infty$. so

$w = \frac{z}{x^2}$

Let us write down $H^0(X, \mathcal{O}(D))$.

$\langle 1, x, z \rangle$ a linear system.

$(1) = 3\infty$
$(x) = s + t + \infty$
$(z) = p + q + r.$



BPF. Also separates points and tangent vectors.

$$\varphi : X \longrightarrow \mathbb{P}^2 \qquad [A:B:C]$$

Note: $\varphi(X)$ satisfies $AC^2 = A^3 + B^3$.

Genus of a plane curve of degree d is

$$\frac{(d-1)(d-2)}{2}.$$

(See next page).

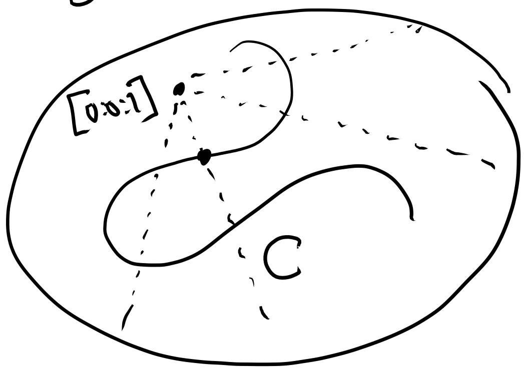
Thm: (Riemann-Roch)

X a compact R.S. K a canonical divisor.

D a divisor on X. of degree d

$$\dim H^0(X, \mathcal{O}(D)) = d - g + 1 + H^0(X, \mathcal{O}(K-D)).$$

What is $g(C)$?



$\mathbb{P}^2 \quad [X:Y:Z]$

$C \xrightarrow{\varphi} \mathbb{P}^1 \quad , L = \mathcal{O}(1) \quad , \quad X, Y.$

Want X, Y not to vanish simult. on C

i.e. $\quad [0:0:1] \notin C.$

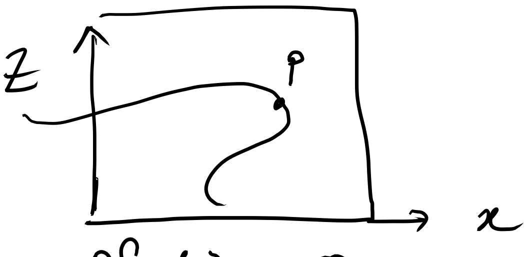
$$\mathbb{P}^2 - \{[0:0:1]\} \longrightarrow \mathbb{P}^1$$
$$[X:Y:Z] \longmapsto [X:Y]$$

Q: What is $\mathrm{Ram}(\varphi)$?

$C \ni p = [A:B:C] \qquad$ assume $B \neq 0$

$$= \Big[\underbrace{\tfrac{A}{B}}_{x} : 1 : \underbrace{\tfrac{C}{B}}_{z}\Big] \in \mathbb{C}^2_{(x,z)}$$

$$\begin{array}{ccl} C \cap \mathbb{C}^2_{(x,z)} & \text{def. by} & f(x,z) = 0 \\ \downarrow \qquad \downarrow & & f(x,z) = F(x,1,z). \\ \mathbb{C} \qquad x. & & \end{array}$$



p is ramified iff $\dfrac{\partial f}{\partial z}(p) = 0.$

More: $\quad \mathrm{Ord}_p(\mathrm{Ram}\,\varphi) = \mathrm{Ord}_p \dfrac{\partial f}{\partial z} = \mathrm{Ord}_p\left(\dfrac{\partial F}{\partial Z}\right)$

So $\mathrm{Ram}(\varphi) = \mathrm{div}\left(\frac{\partial F}{\partial Z}\right)$.

$\hookrightarrow$ section of $\mathcal{O}(d-1)$.

So $\deg \mathrm{Ram}(\varphi) = d \cdot (d-1)$

so $2g_C - 2 = (-2) \cdot d + d(d-1)$

$= d^2 - 3d$

$g_C = \frac{d^2 - 3d + 2}{2}$

$g_C = \frac{(d-1)(d-2)}{2}$

Q: 1) Are all compact R.S. of genus $\frac{(d-1)(d-2)}{2}$ plane curves of degree d?

2) What about R.S. of genus not of this form?

3) What's the "Simplest" way to exhibit a R.S. as a projective curve?