Last time.

- A linear system (V, D) on a compact Riemann surface X yields a closed embedding
$$X \hookrightarrow \mathbb{P}^n$$
iff it separates points and separates tangent vectors.

Let us take $V = H^0(X, D)$ "complete linear system."
We have an inclusion

$$\begin{array}{ccc}
i_p: H^0(X, D-p) & \hookrightarrow & H^0(X, D) \\
f & \longmapsto & f \\
\downarrow & & \downarrow \\
(f) + D - p & \longmapsto & (f) + D \\
E & \longmapsto & E + p.
\end{array}$$

So image of i_p = Sections of $H^0(X, D)$ vanishing at p.
p is a base point of (V, D) iff i_p is surj.
not a base point — " — iff i_p is not a surj.

Note: What is coker (i_p) ?
Say $D = D' + n \cdot p$

$$0 \to H^0(X, D-p) \to H^0(X, D) \to \mathbb{C}$$
$$f \longmapsto \text{coeff of } t^{-n} \text{ in local expansion of } f \text{ at } p.$$

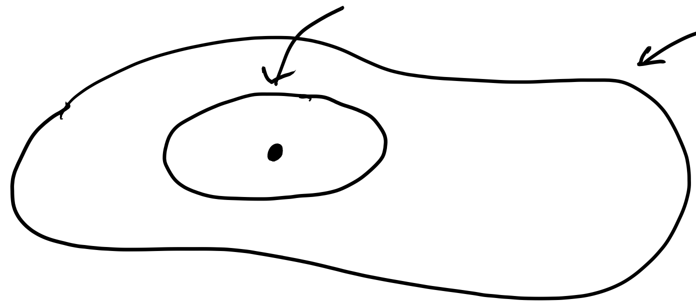
In fact the above arises from

$$0 \to \mathcal{O}(D-p) \xrightarrow{i_p} \mathcal{O}(D) \to \mathbb{C}_p \to 0$$

Let us also interpret i_p in terms of line bundles.

U_1 = disk around p.

U_2 = Compl. of p



$\mathcal{O}(D)$:

$$\mathbb{C}\times U_1 \qquad\qquad \mathbb{C}\times U_2$$
$$(t^n, u) \longleftrightarrow (1, u)$$

$\mathcal{O}(D-p)$:

$$\mathbb{C}\times U_1 \qquad\qquad \mathbb{C}\times U_2$$
$$(t^{n-1}, u) \longleftrightarrow (1, u)$$

$$\mathcal{L}_{D-p} \xrightarrow{i_p} \mathcal{L}_p$$

identity on U_2
mult by t on U_1.

So $(i_p\sigma) = (\sigma) + p$ as expected.

Upshot $V = H^0(X, D)$ is base-point free iff

$$\dim H^0(X, D-p) = \dim H^0(X, D) - 1$$

Let $p, q \in X$. When does V separate p & q?

$$0 \to \mathcal{O}(D-p-q) \to \mathcal{O}(D) \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0$$

$$0 \to H^0(X, D-p-q) \to H^0(X, D) \xrightarrow{\pi} \mathbb{C}^2$$

Separates pts $\Leftrightarrow$ $(1,0)$ & $(0,1)$ lie in the image of π
$\Leftrightarrow$ π is surjective.

$\iff \quad h^0(X, D-p-q) = h^0(X,D) - 2.$

Notice:

$$\cdots \subset H^0(X, D-P_1-P_2) \subset H^0(X, D-P_1) \subset H^0(X,D)$$

$\leq 1 \qquad \leq 1 \qquad \leq 1 \qquad$ dim

so $\dim H^0(X, D-P_1-\cdots-P_n) \geq \dim H^0(X,D) - n$

& if equality holds then

$$\dim H^0(X, D-P_1-\cdots-P_i) = \dim H^0(X, D-P_1-\cdots-P_{i-1}) - 1$$

So if $h^0(X,D-p-q) = h^0(X,D)-2$

$\forall\ p,q \in X$ then $h^0(X,D-p) = h^0(X,D)-1$

$\forall\ p \in X.$

Separates tangent vectors.

$$0 \to \mathcal{O}(D-2p) \to \mathcal{O}(D) \xrightarrow{\pi} \mathbb{C}_{2p} \cong \mathbb{C}_p^2 \to 0$$

$f \mapsto$ coeff of $\underline{t^{-n}}$ & $\underline{t^{-n+1}}$

$$0 \to H^0(X, D-2p) \to H^0(X, \mathcal{O}(D)) \xrightarrow{\pi} \mathbb{C}^2$$

Sep tang vec $\iff (0,1) \in \operatorname{Im} \pi$

No bp. $\iff (1,0) \in \operatorname{Im} \pi$.

so No bp + sep tan vec $\iff \pi$ surj

$\iff h^0(X, D-2p) = h^0(X,D) - 2.$

Conclusion: $(V = H^0(X,D), D)$ yields an embedding iff

$$h^0(X, D-p-q) = h^0(X,D) - 2$$

$\forall\ p,q \in X$, including $p=q$.

Example: $X = \mathbb{P}^1$, $V = H^0(X, \mathcal{O}(d)) = H^0(X, d\cdot p)$

$$h^0(X, \mathcal{O}(d)) = d+1$$

$$h^0(X, d\cdot p - p_1 - p_2) = h^0(X, \mathcal{O}(d-2)) = d-1$$

so V gives an embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$

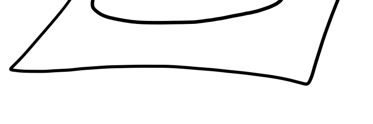
Concretely.

$$x \mapsto [1 : x : x^2 : \dots : x^d]$$

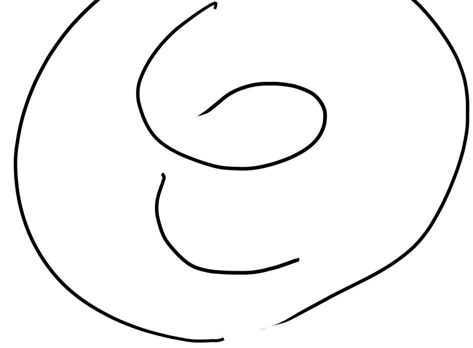
$$[X:Y] \mapsto [Y^d : YX^{d-1} : \dots : X^d]$$

"Rational normal curves".

Line



plane conic



twisted cubic

Luckily, we knew $h^0(X,D)$ for any D on $X = \mathbb{P}^1$

<u>Thm</u> (Riemann-Roch).

Let X be a compact Riemann surface of genus g. and D a divisor of degree d. on X.

Then $$h^0(X,D) = d-g+1 + h^0(X, K-D).$$

K = canonical divisor (class) of X
(Has degree $2g-2$.)

<u>Rem</u>: $\deg(K-D) = 2g-2-d$.
So if $d > 2g-2$, then $H^0(X, K-D) = 0$.

so $h^0(X,D) = d-g+1$.

<u>Conseq</u>: <u>Thm</u> – Let X be a compact R.S. of genus g and D a divisor on X of degree $\geq 2g+1$. Then the complete linear system associated to D gives an embedding

$$X \hookrightarrow \mathbb{P}^n. \qquad (n = d-g).$$

<u>Pf</u>: $h^0(X, D-p-q) = (d-2)-g+1$ by Riemann-Roch, because $d-2 > 2g-2$.

so $h^0(X, D-p-q) = h^0(D) - 2$

$\square$