

**MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES —  
HOMEWORK 1**

1. HYPERELLIPTIC CURVES

- (1) Let  $f(x) \in \mathbb{C}[x]$  be a polynomial of degree  $2n$  without repeated roots. Let  $U \subset \mathbb{C}^2$  be the Riemann surface defined by  $y^2 - f(x) = 0$ . Construct explicitly a compact Riemann surface  $X$  containing  $U$  along with a map  $\pi: X \rightarrow \mathbb{P}^1$ .
- (2) Let  $D \subset \mathbb{C}$  be a small disk that does not contain any zeros of  $f(x)$ . Prove that the preimage  $\pi^{-1}(D)$  is biholomorphic to a disjoint union of two disks. What if  $D$  contains a zero of  $f(x)$ ?
- (3) Compute the Euler characteristic and hence the genus of  $X$ .
- (4) Prove that the field of meromorphic functions on  $X$  is isomorphic to

$$\mathbb{C}(x)[y]/(y^2 - f(x)).$$

2. CYCLIC COVERINGS

- (5) Generalize as much of the above as you can to the curve defined by  $y^n - f(x) = 0$ , where  $f(x) \in \mathbb{C}[x]$  is a polynomial of degree divisible by  $n$  without repeated roots.
- (6) What happens in the analysis above if the degree of  $f(x)$  is not divisible by  $n$ ?

3. COMPLEX TORI

- (7) Let  $X = \mathbb{C}/\Lambda$ , where  $\Lambda \subset \mathbb{C}$  is a lattice. Note that addition induces a (holomorphic) group law on  $X$ . Show that under this law,  $X$  is a divisible group. For a positive integer  $n$ , describe the group  $X[n]$  of  $n$ -torsion points on  $X$ .

4. PLANE CURVES

- (8) Let  $X \subset \mathbb{P}^2$  be a smooth plane curve of degree 1 or 2. Show that  $X$  is isomorphic to  $\mathbb{P}^1$ .
- (9) Let  $C \subset \mathbb{P}^2$  be the Fermat curve, defined by

$$X^d + Y^d + Z^d = 0.$$

By analyzing the ramification of the map  $C \rightarrow \mathbb{P}^1$  given by  $[X : Y : Z] \mapsto [X : Y]$ , find the genus of  $C$ .

5. LINE BUNDLES

- (10) We defined the line bundle  $\mathcal{O}(m)$  on  $\mathbb{P}^1$  in class. Identify the space of global holomorphic sections of  $\mathcal{O}(m)$  with polynomials in  $\mathbb{C}[x]$  of degree at most  $m$ , or equivalently, with homogeneous polynomials of degree  $m$  in  $\mathbb{C}[x, y]$ .