MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES — HOMEWORK 2

(1) Let X be the plane curve defined by

$$Y^2Z = X^3 - XZ^2.$$

  (a) Show that X is a smooth curve.
  (b) Let $p_0 = [0:1:0]$, $p_1 = [0:0:1]$, $p_2 = [1:0:1]$, and $p_3 = [-1:0:1]$. Show that $2p_0 \sim 2p_i$ for each i.
  (c) Show that $p_1 + p_2 + p_3 \sim 3p_0$.

(2) Let X be the compactification of the plane curve

$$y^2 = x^6 - 1$$

constructed as in Homework 1. Let $\pi\colon X \to \mathbb{P}^1$ be the degree 2 map that extends $(x, y) \mapsto x$. Let $\zeta = e^{i\pi/3}$, and set

$$r_i = (\zeta^i, 0), \quad p + q = \pi^*(\infty), \quad s_1 = (0, i), \quad s_2 = (0, -i).$$

  (a) Show that $p + q \sim 2r_i$ for all i.
  (b) Show that $\sum_{i=0}^{5} r_i \sim 3p + 3q$.

(3) Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $2n$. Let X be the compactification of the plane curve

$$y^2 - f(x) = 0$$

constructed as in Homework 1.

  (a) Find $\operatorname{div}(dx)$ on X.
  (b) Find $\operatorname{div}(dy)$ on X.
  (c) Write down all the holomorphic differential forms on X.
  Hint: Use the description of the field of meromorphic functions on X. To check your answer, verify that the forms you wrote span a g_X dimensional vector space.

(4) Let $C \subset \mathbb{P}^2$ be the Fermat curve $X^d + Y^d + Z^d = 0$. Write down all the holomorphic differential forms on C.

You may use that every meremorphic function on C is of the form F/G, where F and G are homogeneous polynomials in X,Y,Z of the same degree and $X^d + Y^d + Z^d$ does not divide G.