MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES — HOMEWORK 3

(1) Compute the cohomology groups $H^i(X, \mathbb{Z})$ using Cech cohomology for $X = \mathbb{P}^1$.

(2) Let L be a vector bundle on X. Establish a bijection between (a) the set of sections of L, and (b) the set of maps of vector bundles from the trivial bundle $\mathbb{C} \times X$ to L. Conclude that a line bundle is trivial if and only if it has a nowhere vanishing section.

(3) Let D be a divisor of degree 0 on a compact Riemann surface X. Show that

$$\dim H^0(X, \mathcal{O}(D)) = \begin{cases} 1 & \text{if } D \sim 0 \\ 0 & \text{otherwise.} \end{cases}$$

(4) Let D be a non-zero effective divisor on a compact Riemann surface X such that

$$\dim H^0(X, \mathcal{O}(D)) = \deg D + 1.$$

Show that $X \cong \mathbb{P}^1$
Hint: Show that there exists $p \in X$ such that $\dim H^0(X, \mathcal{O}(p)) = 2$.

(5) Let X be a compact Riemann surface and $U = X \setminus S$, where S is a finite set. Show that a every holomorphic map $U \to \mathbb{P}^n$ given by homogeneous coordinates which are meromorphic functions on X extends to a holomorphic map $X \to \mathbb{P}^n$.

(6) Let D be a very ample divisor on X. Show that the meromorphic functions in $H^0(X, \mathcal{O}(D))$ separate the points and tangent vectors on X. Conclude that they generate $\mathcal{M}_X$.

(7) Let X be a compact Riemann surface and $S \subset X$ a finite set. Use the Riemann-Roch theorem to show that there exists a meromorphic function on X that is holomorphic on $X \setminus S$.

(8) (Counts as 2 problems) Let U be the Riemann surface in $\mathbb{C}^2$ defined by $y^3 = x^5 - 1$. We know that there exists a unique compact Riemann surface X containing U that extends the map $U \to \mathbb{A}^1$ (given by projecting on the x-line) to a map $\pi : X \to \mathbb{P}^1$. We also know that $X \setminus U$ consists of a finite set of points.
  (a) Show that there is a unique point in $X \setminus U$, call it ∞.
  (b) Find the ramification divisor of π and hence the genus of X.
  (c) Write down the canonical series $H^0(X, K)$.
  (d) Show that the canonical map yields an embedding $X \to \mathbb{P}^3$.
  (e) Show that the canonical image of X lies on a singular quadric hypersurface.

(9) Let D be a divisor on a smooth algebraic curve X of genus g such that $\deg D = 2g - 2$ and $\dim H^0(X, \mathcal{O}(D)) = g$. Show that D is a canonical divisor (that is, $\mathcal{O}_X(D) \cong K_X$).