MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES — HOMEWORK 4

Generic sections and special divisors.

(1) Let (V, D) be a base-point free linear series on a compact Riemann surface X. Show that there exists $\sigma \in V$ such that (σ) is multiplicity free. (This is a special case of something called Bertini's theorem.)

(2) Let D be a divisor and E an effective divisor. Show by induction on $\deg E$ that

$$h^0(D - E) \geq \max(0, h^0(D) - \deg E).$$

Also show that the inequality is sharp—that is, given a D, there exists an E of every degree such that equality holds.

(3) Let D be a divisor on X of degree d. Show that we have

$$h^0(D) \begin{cases} = d - g + 1 & \text{if } d > 2g - 2 \\ \geq d - g + 1 & \text{if } 2g - 2 \geq d \geq g \\ \geq 0 & \text{if } g - 1 \geq d \geq 0. \end{cases}$$

Also show that the inequalities are sharp—that is, there exist D of every degree where equalities hold.

Hint: Write $D = H - E$, where H and E are effective and $\deg H$ is huge.

Remark: A divisor (class) D for which $h^0(D)$ is strictly larger than the bounds above is called *special*. Much of the study of algebraic curves (and their moduli space) involves understanding special divisors on curves.

Quadric surfaces and genus 4 curves.

"Quadric" is a commonly used short-form for "degree 2."

(4) Show that an irreducible quadric hypersurface in $\mathbb{P}^3$ is isomorphic to either

$$X^2 + Y^2 + Z^2 + W^2 = 0$$

or

$$X^2 + Y^2 + Z^2 = 0.$$

(5) Show that a smooth quadric hypersurface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(6) Recall that a line through two points P and Q in $\mathbb{P}^n$ is given parametrically by

$$L = \{uP + vQ \mid [u : v] \in \mathbb{P}^1\}.$$

Use this to describe all the lines on the smooth quadric $X^2 + Y^2 + Z^2 + W^2 = 0$ and the singular quadric $X^2 + Y^2 + Z^2 = 0$.

Let X be a compact Riemann surface of genus 4. Suppose X is not hyperelliptic. Then X is embedded in $\mathbb{P}^3$ by the canonical linear series. In the following problems, use that $X \subset \mathbb{P}^3$ is the intersection of an (irreducible) quadric hypersurface and a cubic hypersurface.

(7) Using geometric Riemann–Roch and the geometry of quadric hypersurfaces from the previous problems, show that there exist exactly two g^1_3's on X if Q is smooth, and exactly one g^1_3 on X if Q is singular.

(8) Suppose X is a compact Riemann surface of genus 4 with two g^1_3's, say D_1 and D_2. Use Riemann–Roch to show that

$$D_1 + D_2 \sim K_X.$$

Similarly, if X has only one g^1_3, say D, then show that

$$2D \sim K_X.$$

Branched covers and monodromy.

(9) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d, defined by $F(X, Y, Z) = 0$. Assume that $[0 : 0 : 1]$ does not lie on X. Consider the projection $C \to \mathbb{P}^1$ given by $[X : Y : Z] \mapsto [X : Y]$. Show that the ramification divisor of C is the zero locus of on C of the homogeneous polynomial $\frac{\partial F}{\partial Z}$. Using Riemann–Hurwitz, conclude that the genus of C is $d(d-1)/2$.

(10) Let C be the Fermat curve

$$X^d + Y^d + Z^d = 0.$$

Consider the projection $\phi\colon C \to \mathbb{P}^1$ that drops the Z coordinate (see (9)). Find $\operatorname{br}\phi \subset \mathbb{P}^1$ and determine the monodromy map

$$\pi_1(\mathbb{P}^1 \setminus \operatorname{br}\phi) \to S_d.$$

(11) Let X be a compact Riemann surface of genus g. Given a finite subset $B \subset X$ of even cardinality, show that that there are 2^{2g} double covers of X with branch divisor B (If B is empty, then one of them will be disconnected).